Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Approximation of $W^{2,2}$ isometric immersions by smooth isometric immersions

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by

Peter Hornung

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Fine level set structure of flat isometric immersions

Peter Hornung

Abstract

A result by Pogorelov asserts that C^1 isometric immersions u of a bounded domain $S \subset \mathbb{R}^2$ into $\mathbb{R}^3$ whose normal takes values in a set of zero area enjoy the following regularity property: The gradient $f := \nabla u$ is 'developable' in the sense that the nondegenerate level sets of f consist of straight line segments intersecting the boundary of S at both endpoints.

Motivated by applications in nonlinear elasticity, we study the level set structure of such f when S is an arbitrary bounded Lipschitz domain. We show that f can be approximated by uniformly bounded maps with a simplified level set structure. We also show that the domain S can be decomposed (up to a controlled remainder) into finitely many subdomains, each of which admits a global line of curvature parametrization.

Contents

1 Introduction

A C^1-regular mapping u from a bounded Lipschitz domain $S \subset \mathbb{R}^2$ into $\mathbb{R}^3$ is called an isometric immersion of S (endowed with the flat metric) if u satisfies the partial differential system

$$\partial_\alpha u \cdot \partial_\beta u = \delta_{\alpha\beta} \text{ for } \alpha, \beta = 1, 2 \tag{1}$$

pointwise in S; here, $\delta_{\alpha\beta}$ is the Kronecker symbol. The classical results from [8] show that if $u \in C^2$ is an isometric immersions of S then the nondegenerate level sets of ∇u are straight line segments intersecting ∂S at both endpoints. In [20], Chapter IX.4, it is shown that the same conclusion remains true if u is only C^1, provided that the Gauss map $\partial_1 u \wedge \partial_2 u : S \to \mathbb{S}^2$ takes values in a set of zero area. To formulate these results precisely, let us define, for any continuous mapping f on S, the set of local constancy

$$C_f = \{x \in S : f \text{ is constant in a neighbourhood of } x\}. \tag{2}$$

Then [8], [20] assert that the gradient $f := \nabla u$ enjoys the property (L) (introduced in [15]), that is:

Through every point $x \in S \setminus C_f$ there exists a line segment $[x] \subset S$ whose endpoints are contained in ∂S, and f is constant on $[x]$. Different line segments do not intersect in S.

More recently, in [14] and [18] it was shown that $W^{2,2}$-regular isometric immersions of S are always C^1, and that their gradient $f = \nabla u$ also satisfies condition (L); their proof of the latter fact uses the integrability of the second derivatives and thus differs from the proof in [20]. In [14], [18] it is also shown that functions $V \in W^{2,2}(S)$ solving the homogeneous Monge-Ampère equation

$$\det \nabla^2 V = 0, \tag{3}$$

belong to $C^1(S)$, and that their gradient $f = \nabla V$ satisfies condition (L). In fact, their assertion about $W^{2,2}$ isometric immersions u follows from this observation because every component of u is a $W^{2,2}$-regular solution of (3).

Motivated by the above results, in the present paper we study continuous mappings $f : S \to \mathbb{R}^P$ (for any $P \in \mathbb{N}$) which satisfy condition (L), but which are otherwise arbitrary. Hence the setting considered here is closer to the hypotheses in [20] than to those in [14], [18], [15] because we make no integrability assumption about the derivatives of f. By the results mentioned earlier, the gradient of any C^1 isometric immersion whose spherical image has vanishing area falls into the framework considered here; the same is true for $W^{2,2}$ isometric immersions and for $W^{2,2}$ solutions of (3). A generalization of the developability of $W^{2,2}$-solutions of (3) to a more general class of functions was recently obtained in [13], cf. also [12]. The corresponding solutions, however, do not quite fall into our framework because the mappings arising there can be discontinuous.

Regarding elasticity, some recent interest in $W^{2,2}$ isometric immersions was stimulated by the rigorous derivation of Kirchhoff's plate theory in [7], cf. also [19]. This theory assigns to every deformation u of a two-dimensional reference configuration $S \subset \mathbb{R}^2$ (i.e., the mid-plane of some asymptotically thin film) into $\mathbb{R}^3$ the pure bending energy

$$\mathcal{E}_K(u) = \begin{cases} \int_S |\nabla^2 u|^2 & \text{if } u \in W^{2,2}(S;\mathbb{R}^3) \text{ satisfies (1)} \\ +\infty & \text{otherwise.} \end{cases} \tag{4}$$

The class of $W^{2,2}$ isometric immersions therefore arises naturally as the set of finite energy deformations in Kirchhoff's plate theory; they are characterized by the condition that, locally, they preserve angles and distances. $W^{2,2}$ isometric immersions also play an important role in the derivation of related theories, cf. e.g. [24] and [4].

Regarding geometry, condition (L) amounts to the assertion that the surface $u(S)$ consists of planar pieces (namely $u(C_f)$) and developable pieces (the remainder of $u(S)$). Both parts can consist of infinitely many connected components.

There is a large body of literature on flat surfaces (i.e. surfaces with zero Gauss curvature). In introductory texts as [2] it shown that, away from certain singular sets, a developable surface is a union of cylinders, cones and tangent developables. More recently, developable surfaces (or certain generalizations) have been studied, e.g., in [23], [25], [26] and [16]; see also the references cited therein.

The emphasis of the present paper lies on the properties of the level sets of ∇u in S and not on the properties of the image $u(S)$ as a subset of $\mathbb{R}^3$, although, of course, these two are related. The questions addressed here are closest to the paper [8] and to some results in [14] and [18] for convex domains S. Our main contribution is a fairly precise description of the level set geometry on arbitrary Lipschitz domains S. The results and tools developed here are essential in [10], where it is shown that $W^{2,2}$ isometric immersions can be approximated by C^∞ isometric immersions (thus extending earlier results from [18]) and in [9], where minimizers of (4) are studied.

The present paper consists of two parts. In the first part (starting in Section 3) we study the geometry of the set C_f. On convex domains S the geometry of the connected components of C_f is fairly simple: They are convex polygons (with possibly infinitely many vertices), see Theorem A in Section 9 of [8] and Proposition 2.30 in [14]. For nonconvex domains this is no longer true in general. On domains which are not even simply connected, the set C_f is even less constrained, mainly due to the topological fact that a line segment in S whose endpoints lie on different connected components of ∂S does not disconnect S.

In the second part (starting in Section 5) we study the set $S \setminus C_f$ containing the nondegenerate level sets of f. This set can be locally described by (generalized) lines of curvature Γ. Their defining property is that they run perpendicular to the level set lines $[x]$. A mapping f satisfying condition (L) strongly interacts with the boundary ∂S. The situation on nonconvex domains is more subtle than for convex domains, mainly due to the fact that the segments $[x]$ can intersect ∂S tangentially. Also a nontrivial topology of S leads to further effects not present on simply connected domains.

Apart from the above 'descriptive' results, we also obtain two constructive results. The first one, Theorem 3, asserts that every continuous mapping satisfying condition (L) can be approximated by mappings (arising from the original one by a suitable truncation) which are 'finitely developable'; roughly speaking, these are mappings for which C_f and its complement consist of finitely many connected components. This reduces the complexity of other approximating constructions, cf. [10]. The approximants constructed in Theorem 3 happen to belong to L^∞; as a corollary we thus recover Theorem 2 from [15] (but not their Remark 12).

The second constructive result is Theorem 4, which shows that the domain S can be decomposed into a finite number of natural subdomains which are compatible with f in a suitable sense plus (possibly infinitely many) subdomains which are

localized — both geometrically and topologically — near the boundary ∂S. The latter theorem, as well as the results in [10] and [9] rely heavily on the descriptive results obtained in the earlier sections of this paper.
The techniques used in this paper are quite basic. Apart from some analysis we mainly use topological facts about subsets of $\mathbb{R}^2$.

1.1 The set C_f and a truncation theorem

The purpose of this subsection and the following one is to describe the key results of this paper. In order to keep notation at a minimum, most of them will be presented as simplified versions of the actual statements proven later. For simplicity, throughout this introduction we assume that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain.

The basic object of study of this paper are countably developable mappings. This is how we call continuous mappings satisfying condition (L). In order to make this definition analytically more accessible, we introduce the directed distance function

$$\nu^S(x,\mu) = \inf\{\theta > 0 : x + \theta\mu \notin S\} \text{ for all } (x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\}), \tag{5}$$

which also plays a critical role in the related papers [10], [9]. A continuous mapping f is countably developable precisely if f is 'developable' on $S \setminus C_f$ in the following sense:
There exists a vector field $q_f : S \setminus C_f \to \mathbb{S}^1$ such that, denoting by $[x]$ the open line segment with endpoints

$$x \pm \nu^S(x, \pm q_f(x))q_f(x),$$

the following are satisfied:

$$x, y \in S \setminus C_f \implies \text{ either } [x] \cap [y] = \emptyset \text{ or } [x] = [y]. \tag{6}$$

$$x \in S \setminus C_f \implies f = f(x) \text{ on } [x]. \tag{7}$$

For the rest of this Introduction $f \in C^0(S;\mathbb{R}^P)$ denotes a countably developable mapping. Clearly the developability of f on $S \setminus C_f$ will have strong implications about the geometry of the set C_f.

In the first part of this paper we will study the set C_f, extending earlier results from [8] and [14]. Our contribution is to study this set, in more detail than in [8], on domains which are merely Lipschitz. In order to motivate the detailed analysis of the set C_f carried out in this paper, we begin by stating our first main result, the truncation Theorem 1. Its proof makes essential use of detailed knowledge about C_f. Before stating this theorem, however, we first need another result and two definitions.
Since C_f is open, it consists of countably many connected components U. For the moment, let us think of each such U as being a polygon (with possibly infinitely many vertices) whose relative boundary $S \cap \partial U$ consists of segments of the form $[x]$; this will be shown in Proposition 3 below. It will be useful single out those connected components U of C_f for which $S \cap \partial U$ consists of at least three connected components (i.e. segments of the form $[x]$). Denote by $\hat{C}_f$ the union of all connected components U of C_f for which this is the case. The following key result shows why the set $\hat{C}_f$ is natural; cf. Proposition 9 for more details.

Proposition 1 *There exists an extension of q_f to $S \setminus \hat{C}_f$ (which is again denoted q_f) such that (6, 7) remain true with $\hat{C}_f$ instead of C_f.*

Remarks.

1. Proposition 1 shows that f is developable on connected components of U for which $S \cap \partial U$ consists of at most two segments $[x]$. A similar result is true for convex domains, cf. [14]; however, the proof for nonconvex sets is more subtle.

2. We will see (cf. Lemma 4) that q_f is uniquely determined on $S \setminus C_f$ modulo identification of antipodal points. This is no longer true for its extension to $S \setminus \hat{C}_f$. But this non-uniqueness is irrelevant in what follows.

3. The conclusion of Proposition 1 is the underlying reason why in the regularity analysis of [9] the relative boundary $S \cap \partial\hat{C}_f$ occurs in the singular set.

For general countably developable mappings f, the set $\hat{C}_f$ (and hence C_f) may consist of infinitely many connected components, and each of them can be bounded by infinitely many line segments. For instance, if S is the unit disk then $\hat{C}_f$ can consist, e.g., of one single connected component which is a polygon with infinitely many vertices on ∂S. The connected components of $\hat{C}_f$ can even accumulate at a line segment in S, even if S is a disk. This fact makes explicit constructions difficult. A better class of mappings is defined as follows:
A countably developable mapping f on S is said to be *finitely developable* if $\hat{C}_f$ consists of finitely many connected components U and for each of them the set $S\cap\partial U$ also consists of only finitely many connected components. Our first constructive result is the following theorem; a more precise version is Theorem 3 below.

Theorem 1 *For all $\delta > 0$, there exists a finitely developable mapping $f_\delta \in C^0(S;\mathbb{R}^P)$ such that $f_\delta = f$ on $\{x \in S : \mathrm{dist}_{\partial S}(x) \geq \delta\}$, and f_δ is constant on each connected component of $\{x \in S : f_\delta(x) \neq f(x)\}$.*

Remarks.

1. Theorem 1 is a truncation result. It is easy to see that each f_δ is uniformly bounded on S, cf. Theorem 3. Thus we recover Theorem 2 in [15] (but not their Remark 12). However, this is not needed in here, nor in [10]. What will be needed is that f_δ is finitely developable.

2. In terms of isometric immersions $u : S \to \mathbb{R}^3$, Theorem 1 can be used to show that $u(S)$ can be approximated by surfaces consisting of finitely many planar regions and finitely many developable regions; cf. [10].
This is the initial step in the construction of smooth isometric immersions in [10]. Generally, constructions starting from a countably developable mapping f will be easier to carry out if it is known that f is finitely developable.

In order to show that the simple truncation argument employed to prove Theorem 1 is enough to obtain a finitely developable mapping f_δ, one has to understand the overall stucture of C_f: The key ingredient in the proof of Theorem 1 is the following proposition, which is a simplified version of Proposition 8 below.

Proposition 2 *For every $\delta > 0$ there exist only finitely many connected components U of C_f which have more than two boundary segments $[x] \subset S \cap \partial U$ with $\mathcal{H}^1([x]) \geq \delta$.*

In other words, all except finitely many components U 'almost' belong to the set $S \setminus \hat{C}_f$ on which the mapping q_f satisfying (6, 7) is well-defined by virtue of Proposition 1.

In order to prove a result like Proposition 2 one needs to understand well the structure of each component U of C_f. The main result in this regard is the following one; a precise and more detailed version is Proposition 7 below.

Proposition 3 *Every connected component U of C_f satisfies the following* **Condition** (B_f)*:*

$$x \in S \cap \partial U \implies [x] \subset S \cap \partial U. \tag{8}$$

Moreover, U is locally on one side of each segment $[x] \subset S \cap \partial U$.

A result like Proposition 3 is not very useful unless more precise information is extracted from (8). We will see that Condition (B_f) is the natural compatibility condition with f for any subdomain $U \subset S$. In fact, we will encounter several other sets arising naturally in this context and which also satisfy condition (B_f). Moreover, condition (B_f) alone has several consequences for the regularity of the set in question: In Lemma 5 below we derive properties of arbitrary subdomains U of S satisfying condition (B_f). Since the statement of that lemma is slightly technical, here we content ourselves with noting some of its consequences:

Lemma 1 *Let $U \subset S$ be a subdomain satisfying condition (B_f). Then U has finite perimeter, and the relative boundary $S \cap \partial U$ consists of countably many disjoint segments of the form $[x]$. Moreover, if $\partial_1 S$ and $\partial_2 S$ are two connected components of ∂S then there are at most two segments $[x]$ in $S \cap \partial U$ with the property that $[x]$ has one endpoint in $\partial_1 S$ and the other one in $\partial_2 S$.*

The link between Proposition 3 and Lemma 1 on one hand and Proposition 2 on the other hand is the observation that all except finitely many connected components of C_f are (topologically) 'squeezed' (cf. Lemma 3 for a precise notion on nonconvex domains) between two of their boundary segments.

1.2 The set $S \setminus C_f$ and a decomposition theorem

When f is the gradient of a flat isometric immersion $u : S \to \mathbb{R}^3$ then the set $S \setminus C_f$ corresponds to the set where the surface $u(S)$ is not planar, i.e., there exists one principal curvature which differs from zero. On such a region there exists a well-defined line of curvature $\Gamma : [0, T] \to S \setminus C_f$; as usual we identify $u(S)$ with S via u. The line of curvature Γ is characterized by the condition

$$\Gamma'(t) = -q_f^{\perp}(\Gamma(t)) \text{ for all } t \in [0, T]. \tag{9}$$

For general countably developable f this definition makes sense as well, even if Γ takes values in the larger set $S \setminus \hat{C}_f$, cf. Proposition 1. Curves taking values in this set and satisfying (9) — plus an additional technical condition which amounts to Γ being short compared to its distance from ∂S — will be called f-integral curves. Continuing the analogy with surfaces, observe that a line of curvature $\Gamma : [0, T] \to S$ naturally induces the change of variables $\Phi_\Gamma : [0, T] \times \mathbb{R} \to \mathbb{R}^2$ given by

$$\Phi_\Gamma(s, t) = \Gamma(t) + sN(t)$$

between line of curvature coordinates (s, t) and the usual coordinates $x = (x_1, x_2)$. The mapping Φ_Γ is a natural tool to study f, because by (9)

$$f(\Phi(s, t)) = \Gamma(t) \text{ for all } t \in [0, T] \text{ and all } s \in (s_\Gamma^-(t), s_\Gamma^+(t)); \tag{10}$$

here,

$$s_\Gamma^\pm(t) := \pm\nu^S(\Gamma(t), \pm N(t))$$

denote the (signed) directed distance along Γ. The intersection of the subgraph of s_Γ^+ with the epigraph of s_Γ^-, that is, the set

$$M_{s_\Gamma^\pm} := \bigcup_{t \in (0,T)} (s_\Gamma^-(t), s_\Gamma^+(t)) \times \{t\} \tag{11}$$

is the natural domain of Φ_Γ. Its image

$$[\Gamma(0,T)] := \Phi_\Gamma(M_{s_\Gamma^\pm}) \subset S$$

is the patch of S which is parametrized by Γ via Φ_Γ.

Our second main 'constructive' result is the following decomposition theorem, cf. Theorem 4 for a much more detailed version. It shows that the domain S can be partitioned, up to a small remainder which is topologically and geometrically close to ∂S, into finitely many subdomains $U \subset \hat{C}_f$ on which f is constant and finitely many subdomains which are of the form $[\Gamma(0,T)]$ for some f-integral curve Γ.

Theorem 2 *Assume that f is finitely developable and denote by*

$$V_1, ..., V_M \text{ the connected components of } \hat{C}_f.$$

Then, for all $\delta > 0$ there is $N \in \mathbb{N}$ with $N \geq M$, and there exist f-integral curves $\Gamma^{(M+1)} \in W^{2,\infty}([0,T_{M+1}]; S \setminus \hat{C}_f)$, ..., $\Gamma^{(N)} \in W^{2,\infty}([0,T_N]; S \setminus \hat{C}_f)$, such that, setting

$$V_k = [\Gamma^{(k)}(0,T_k)] \text{ for } k = M+1, ..., N,$$

the following are true:

(i) **Covering.** *We have*

$$\{x \in S : \operatorname{dist}_{\partial S}(x) > \delta\} \subset \operatorname{int}\left(\bigcup_{k=1}^{N} \bar{V}_k\right), \tag{12}$$

and the set on the right is a subdomain of S satisfying condition (B_f).

(ii) **Disjoint interiors:** *Whenever $j \neq k$ then*

$$V_j \cap V_k = \emptyset \ (j,k \in \{1, ..., N\}).$$

(iii) **Closures intersect nicely:** *If $j \neq k$ then the set $\bar{V}_j \cap \bar{V}_k \cap S$ consists of at most finitely many segments of the form $[x]$, and $\mathcal{H}^1([x]) \geq \delta$ for each of them.*

Remarks.

1. A consequence of Part (i) is that the complement of the right-hand side of (12) consists of countably many (open, disjoint) connected components which are not only geometrically but also topologically close to ∂S; cf. Theorem 4 for details.

2. Theorem 2 will be used in [10] to construct a global approximant of a given countably developable mapping by gluing together local approximants obtained on sets of the form $[\Gamma(0,T)]$. In this context, the finite developability hypothesis on f made in Theorem 2 is justified by Theorem 1.

3. Part (iii) is crucial. It asserts that the subdomains touch each other in a well controlled way. Theorem 4 even shows that such common segments $[x]$ will always be (essentially) of the form $[\Gamma^{(k)}(T_k)]$ or $[\Gamma^{(k)}(0)]$. This shows that local modifications of f (on each V_k) need only agree with f on $[\Gamma^{(k)}(T_k)]$ and on $[\Gamma^{(k)}(0)]$ in order to give rise to a well-defined mapping, and this requirement is quite easy to fulfill, cf. [10] or [9].

Part (ii) of Theorem 2 is easy to arrange, but part (iii) requires a careful analysis of sets of the form $[\Gamma(0,T)]$. That will be the main issue in this section. Lemma 3.6 in [18] asserts that if S is convex then the set $[\Gamma(0,T)]$ is convex as well. The situation is more complex if S is an arbitrary Lipschitz domain. A first step in understanding this more general situation is the following simplified version of Proposition 16 below:

Proposition 4 *Let Γ be an f-integral curve. Then Φ_Γ is injective on $M_{s_\Gamma^\pm}$, and $[\Gamma(0,T)]$ satisfies condition (B_f), i.e., if $x \in S\cap\partial[\Gamma(0,T)]$ then $[x] \subset S\cap\partial[\Gamma(0,T)]$.*

The injectivity of Φ_Γ is easily seen to follow from (6): It is equivalent to the assertion that

$$[\Gamma(t_1)] \cap [\Gamma(t_2)] = \emptyset \text{ for unequal } t_1, t_2 \in [0,T]; \tag{13}$$

here,

$$[\Gamma(t)] := \{\Gamma(t) + sN(t) : s \in (s_\Gamma^-(t), s_\Gamma^+(t))\}.$$

We will see that the global condition (13) implies the following analytically more accessible, local, condition:

$$1 - s_\Gamma^\pm(t)\kappa(t) \geq 0 \text{ for almost every } t \in (0,T); \tag{14}$$

here, $\kappa := \Gamma'' \cdot N$ is the curvature of the arclength parametrized curve Γ.

In order to be a candidate for an f-integral curve for some developable mapping f, a given curve Γ has to satisfy the global condition (13). In order to study curves satisfying (9) for some arbitrary countably developable mapping f, we will therefore isolate conditions (13, 14) and study arbitrary curves taking values in S (regardless of f) and satisfying (13) or (14). For the rest of this Introduction $\Gamma \in W^{2,\infty}([0,T];S)$ denotes an arclength parametrized curve that is otherwise arbitrary. The basic result about $[\Gamma(0,T)]$ and the change of variables Φ_Γ is this:

Proposition 5 *The set $M_{s_\Gamma^\pm}$ defined in (11) is open, and we have:*

(i) If Γ satisfies (14) then $\Phi_\Gamma|_{M_{s_\Gamma^\pm}}$ is an open mapping. In particular, $\Phi_\Gamma(M_{s_\Gamma^\pm})$ is open and

$$\partial[\Gamma(0,T)] \subset \Phi_\Gamma(\partial M_{s_\Gamma^\pm}). \tag{15}$$

(ii) If Γ satisfies (13) then it satisfies (14) and Φ_Γ^{-1} is locally Lipschitz on $[\Gamma(0,T)]$.

(iii) If Γ satisfies (13) and $1 - s_\Gamma^\pm\kappa$ is essentially bounded from below by a positive constant, then the change of variables $\Phi_\Gamma : M_{s_\Gamma^\pm} \to [\Gamma(0,T)]$ is globally Bilipschitz.

Remarks

1. Openness of $M_{s_\Gamma^\pm}$ will follow from the semicontinuity of $s_\Gamma^\pm$ mentioned below.

2. The implication (13) $\implies$ (14) in Proposition 5 (ii) is a basic fact which is extensively used in [9] and in [10].

3. The converse implication (14) $\implies$ (13) is false in general. In Proposition 13 below we will see: If Γ satisfies (16) then (13) is equivalent to (14) plus a natural extra assumption.

4. Part (iii) is related to the existence of an extension of q_f to a domain containing the closure of $[\Gamma(0,T)]$; cf. Proposition 11. This, in turn, will be used in [10] to construct isometric extensions of a given isometric immersion.

Formula (15) and the definition of $M_{s_\Gamma^\pm}$ relate the jump sets of the functions $s_\Gamma^\pm$ to the relative boundary $S \cap \partial[\Gamma(0,T)]$; such a relation is intuitively clear. In order to prove Theorem 2 (iii) we need enough information about $S \cap \partial[\Gamma(0,T)]$. In view of (15) this amounts to understanding the regularity of $s_\Gamma^\pm$. We will obtain three basic regularity results about these functions. The first two follow immediately from corresponding facts about ν^S and the Lipschitz continuity of Γ and N, but the third one hinges on condition (13) and is false for general curves Γ.
In order to state the relevant regularity properties of ν^S we make the following definition: A pair $(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is said to be transversal if μ is not parallel to the tangent to ∂S at the point $x + \nu(x,\mu)\mu \in \partial S$; cf. Definition 5 for a precise definition on Lipschitz domains. Observe that if S is convex then every pair $(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is transversal, so the effects of non-transversality are genuine to nonconvex domains. The regularity properties we will need are:

- ν^S is lower semicontinuous on $S \times (\mathbb{R}^2 \setminus \{0\})$, cf. Lemma 11.
- ν^S is locally Lipschitz on the (open) set of transversal pairs $(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$, cf. Lemma 12.

The first assertion implies that $\pm s_\Gamma^\pm$ are lower semicontinuous, i.e., that the positive function s_Γ^+ is lower semicontinuous and that the negative function s_Γ^- is upper semicontinuous. The second assertion implies that $s_\Gamma^\pm$ are Lipschitz on $(0,T)$ provided that

$$(\Gamma(t), \pm N(t)) \text{ are transversal for all } t \in [0,T]. \tag{16}$$

This can be combined with (15) and with (13) to prove the following topological fact, which is part of Proposition 12:

Proposition 6 *If $\Gamma : [0,T] \to S$ satisfies (13) and (16) then*

$$S \cap \partial[\Gamma(0,T)] = [\Gamma(0)] \cup [\Gamma(T)]. \tag{17}$$

Remarks.

1. If S is convex then (16) is always true, so on convex domains (17) is true for any f-integral curve.
2. Formula (17) is extensively used in [10] and in [9]. In general, it is very useful when making local modifications of f on $[\Gamma(0,T)]$. In fact, if (17) is satisfied then these modifications must only agree with f on $[\Gamma(0)] \cup [\Gamma(T)]$.
3. In general, if (16) fails then (17) is false, too. Instead, the set $S \cap \partial[\Gamma(0,T)]$ can then consist of infinitely many segments $[x]$; this is illustrated in Figure 4 (left).
   Unfortunately, condition (16) is genuinely local: A given curve Γ cannot, in general, be decomposed into finitely many subcurves satisfying (16) on $(0,T)$.

While the (local) Lipschitz regularity of $s_\Gamma^\pm$ was sufficient in [9], for a result like Theorem 2 one needs also global regularity of $s_\Gamma^\pm$, even when (16) is violated. This is because the curves $\Gamma^{(k)}$ from Theorem 2 in general do not satisfy (16), cf. the Remark to Proposition 6. So the set $S \cap \partial[\Gamma^{(k)}(0,T_k)]$ will strictly contain $[\Gamma^{(k)}(0)] \cup [\Gamma^{(k)}(T_k)]$. Unfortunately, the (global) lower semicontinuity of $s_\Gamma^\pm$ is not enough for our purposes.
This leads us to the third regularity result about $s_\Gamma^\pm$ announced earlier. The key result in this regard is Lemma 15. It asserts that if Γ satisfies (13) then the mappings

$$t \mapsto \Gamma(t) + s_\Gamma^\pm(t) N(t)$$

sweep the boundary ∂S in a monotone fashion. A consequence of this is the following corollary (whose conclusion is generally false if (13) is violated):

Corollary 1 *If* Γ *satisfies (13) then* $s_\Gamma^\pm \in BV(0,T)$.

Combining this corollary with formula (15), one can prove a result like Theorem 2 (iii).

Notation. We frequently identify $\mathbb{R}^2$ with the subspace $\mathbb{R}^2 \times \{0\}$ of $\mathbb{R}^3$. The standard basis vectors in $\mathbb{R}^k$ are denoted by e_i. For $\varepsilon > 0$ and $x \in \mathbb{R}^n$ we denote by $B_\varepsilon(x)$ the open ball of radius ε centered at x and for $X \subset \mathbb{R}^n$ we set $B_\varepsilon(X) = \bigcup_{x\in X} B_\varepsilon(x)$. For $v \in \mathbb{R}^2$ set $v^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$. For $x, y \in \mathbb{R}^2$, the notation (x, y) denotes the open segment with endpoints x and y; the notation $[x, y)$ and $[x, y]$ is to be understood accordingly. By $\mathcal{C}(X;U)$ we denote the connected component of X that contains the connected set U, and if $U = \{v\}$ then we set $\mathcal{C}(X;v) = \mathcal{C}(X;\{v\})$. For a mapping $f : A \to B$ and for $X \subset A$ we set $f(X) = \{f(x) : x \in X\}$. The Hausdorff distance between two bounded sets K_1, K_2 of $\mathbb{R}^n$ is defined by $d_{\mathcal{H}}(K_1, K_2) = \sup(\mathrm{dist}_{K_1}(K_2) \cup \mathrm{dist}_{K_2}(K_1))$. By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure, and for $A \subset \mathbb{R}^n$ we denote its n-dimensional Lebesgue measure by $|A|$ or by $\mathcal{L}^n(A)$. If $g : X \to Y$ and $f : Y \to Z$ we will write $f(g)$ rather than $f \circ g$ to denote their composition. We will write $*$ to denote $+$ or $-$ and $⨍$ to denote an average.

Finally, one remark about the formulation of claims: The hypothesis 'Let A, B, C and assume that $\delta > 0$ is small enough' means that there exists $\delta_0 > 0$, depending only on A, B and C, such that the conclusion is true for all $\delta \in (0, \delta_0)$.

2 Topological preliminaries

A subset $S \subset \mathbb{R}^2$ is called a domain if it is open and connected. We call a subset γ of $\mathbb{R}^2$ a Jordan arc if there is an injective mapping $\tilde{\gamma} \in C^0([0,1];\mathbb{R}^2)$ such that $\gamma = \tilde{\gamma}\big((0,1)\big)$. The points $\gamma^- = \tilde{\gamma}(0)$ and $\gamma^+ = \tilde{\gamma}(1)$ are called endpoints of γ. Notice that $\gamma^+ \neq \gamma^-$ and that $\{\gamma^+, \gamma^-\} \cap \gamma(0,1) = \emptyset$. If no confusion can arise we will identify $\tilde{\gamma}$ with its image γ.

A closed Jordan curve $\gamma \subset \mathbb{R}^2$ is a set which is homeomorphic to $\mathbb{S}^1$. If γ is a closed Jordan curve then the Jordan curve Theorem [17] asserts that $\mathbb{R}^2 \setminus \gamma$ consists of precisely two connected components, a bounded one (denoted $U_b(\gamma)$) and an unbounded one (denoted $U_\infty(\gamma)$). We will frequently use this theorem implicitly. A domain whose boundary consists of a single closed Jordan curve is called a Jordan domain.

We call a domain S continuous if it can be written in the form

$$S = U_b(\partial_0 S) \cap \bigcap_{i=1}^{N_S} U_\infty(\partial_i S). \tag{18}$$

Here, $N_S \in \mathbb{N}$ and $\partial_k S$ ($k = 0, ..., N_S$) are pairwise disjoint closed Jordan curves with $\partial_i S \subset U_b(\partial_0 S)$ and $U_b(\partial_i S) \cap U_b(\partial_j S) = \emptyset$ for all $i, j = 1, ..., N_S$. Boundedness of S is trivial, and connectedness follows e.g. from Theorem 16.2 in Chapter V of [17] and from disjointness of the $\partial_k S$. Notice that the boundary ∂S consists of $N_S + 1$ connected components: one outer component $\partial_0 S$ and N_S inner components $\partial_1 S, ..., \partial_{N_S} S$.

Since $\partial_i S \subset U_b(\partial_0 S)$, we also have $\overline{U_b(\partial_i S)} \subset U_b(\partial_0 S)$, $i = 1, ..., N_S$. This follows from Remark 8 in the appendix. Intuitively, S is a domain with 'outer boundary' $\partial_0 S$ and with N_S 'holes', each of which has boundary $\partial_i S$, where $i = 1, ..., N_S$.

A domain S is called a Lipschitz domain if its boundary can be covered with finitely many open disks such that for each disk there exist local coordinates in which the intersection of S with this disk is given by the epigraph of a Lipschitz function. Bounded Lipschitz domains are continuous domains. For a more precise definition

of Lipschitz domains we refer to Section 4.2 in [5].
For a set $X \subset \mathbb{R}^2$, $\mu \in \mathbb{R}^2 \setminus \{0\}$ and $x \in X$ we define

$$[x]_\mu^X = \mathcal{C}\big((x + (\operatorname{span}\mu)) \cap X; x\big).$$

If it is clear from the context to what set X we are referring to (in particular, if $X = S$), then we will omit the superindex X. Observe that $[x]_\mu^S$ is just the open segment with endpoints $x \pm \nu^S(x, \pm\mu)\mu$, where the notation is as in the introduction. In what follows we will often make use of the fact that if $l_n \subset \mathbb{R}^2$ are line segments with $d_{\mathcal{H}}(l_n, l) \to 0$ then l is a (possibly degenerate) line segment as well. Given two points x^+ and $x^- \in \mathbb{R}^2$ we denote by $[x^+, x^-]$ the closed and by (x^+, x^-) the open line segment with endpoints x^+ and x^-.

Lemma 2 *Let $S \subset \mathbb{R}^2$ be a bounded domain, let $\lambda, \mu \in \mathbb{S}^1$ and $x, y \in S$. Then the following hold:*

(i) The endpoints $x^\pm$ of $[x]_\mu$ and $y^\pm$ of $[y]_\lambda$ can be labelled such that $[x]_\mu = (x^-, x^+)$, $[y]_\lambda = (y^-, y^+)$ and $|x^\pm - y^\pm| < 3d_{\mathcal{H}}([x]_\mu, [y]_\lambda)$.

(ii) If $\mu_n \to \mu$ in $\mathbb{S}^1$ and $x_n \to x$ in S then $\sup \operatorname{dist}_{[x_n]_{\mu_n}}([x]_\mu) \to 0$. In particular, $\operatorname{dist}_{[x]_\mu}(y) \geq \limsup_{n\to\infty} \operatorname{dist}_{[x_n]_{\mu_n}}(y)$ for all $y \in \mathbb{R}^2$.

(iii) If S is convex and if $\mu_n \to \mu$ in $\mathbb{S}^1$ and $x_n \to x$ in S then $d_{\mathcal{H}}([x_n]_{\mu_n}, [x]_\mu) \to 0$.

(iv) Let $x_n \to x$ in S and let $\mu_n, \mu \in \mathbb{S}^1$. If $[x_n]_{\mu_n} \cap [x]_\mu = \emptyset$ for all n, then $\mu_n \to \mu$ in the projective space $\mathbb{P}^1$, i.e. there are $\sigma_n \in \{+1, -1\}$ such that $\sigma_n \mu_n \to \mu$. In particular, if S is convex, then $d_{\mathcal{H}}([x_n]_{\mu_n}, [x]_\mu) \to 0$.

(v) If S is continuous, and with notation as in (i): There is $\varepsilon_0 > 0$ (depending only on S) such that if $d_{\mathcal{H}}([x]_\mu, [y]_\lambda) < \varepsilon_0$, then there exist i, j such that x^-, $y^- \in \partial_i S$, and x^+, $y^+ \in \partial_j S$.

(vi) If S is continuous, and with notation as in (i): There is $\varepsilon_1 > 0$ (depending only on S) such that if $\sup \operatorname{dist}_{\partial S}([x]_\mu) < \varepsilon_1$ then both endpoints x^+ and x^- of $[x]_\mu$ lie in the same connected component of ∂S.

(vii) If S is continuous and $[x]_\mu$ has both endpoints in the same connected component of ∂S then $S \setminus [x]_\mu$ consists of exactly two connected components S_x^1 and S_x^2. If $r > 0$ is such that $B_r(x) \subset S$ then $B_r(x) \setminus [x]_\mu$ consists of two open half-balls B_x^1 and B_x^2 such that $B_x^1 \subset S_x^1$ and $B_x^2 \subset S_x^2$.

Remark. Regarding Lemma 2 (iv), notice that the condition $\sup \operatorname{dist}_{\partial S}([x]_\mu)$ being small does not imply that $\mathcal{H}^1([x]_\mu)$ is small: Consider e.g. a square S and $[x]_\mu$ parallel and very close to one of its sides.

Proof. To prove (i) write $l_x = [x^- x^+]$ and $l_y = [y^- y^+]$ and notice that, setting $\varepsilon = d_{\mathcal{H}}(l_x, l_y)$ we have $l_x \subset \bar{B}_\varepsilon(l_y)$, so there exist $y'_-, y'_+ \in l_y$ such that $|x^\pm - y'_\pm| \leq \varepsilon$. Setting $l'_y = [y'_- y'_+]$, by convexity of $\bar{B}_\varepsilon(l'_y)$ we have $l_x \subset \bar{B}_\varepsilon(l'_y)$. Hence $l_y \subset \bar{B}_\varepsilon(l_x) \subset \bar{B}_{2\varepsilon}(l'_y)$, so since $l'_y \subset l_y$, after possibly swapping y^+ and y^- we conclude that $y^* \in \bar{B}_{2\varepsilon}(y'_*)$. Hence $y^* \in \bar{B}_{3\varepsilon}(x^*)$, $* = +, -$, and (i) is proven. Item (v) follows from (i) and the fact that the connected components of ∂S have positive distance from each other.

To prove (iii) let S be convex. Since ∂S is compact, after passing to subsequences, and writing $[x_n]_{\mu_n} = (x_n^- x_n^+)$, we have $x_n^* \to: y^* \in \partial S$ for $* = +, -$. Hence with $Y = [y^- y^+]$, by convexity there is a sequence $\varepsilon_n \downarrow 0$ such that $x_n \in B_{\varepsilon_n}(Y)$ for all n. Hence $x \in Y$. Summarizing, $y^\pm \in \partial S$ and $x \in [y^- y^+] \cap S$. Since S is convex, by Theorem 6.1 in [22] this implies that $(y^- y^+) \subset S$. So by maximality of

$[x]_\nu$ we have $[x]_\nu = (y^- y^+)$ for $\nu := \frac{y^+ - y^-}{|y^+ - y^-|}$. Notice that $\mathcal{H}^1(Y) = |y^+ - y^-|$ is positive because $x \in (y^- y^+)$, so $|x_n^+ - x_n^-| \geq \frac{\mathcal{H}^1(Y)}{2} > 0$ for large n. Since μ_n as well as the $x_n^\pm$ converge, there is $\sigma \in \{1, -1\}$ such that for large n we have

$$\mu_n = \sigma \frac{x_n^- - x_n^+}{|x_n^- - x_n^+|}. \tag{19}$$

The right-hand side converges to $\sigma \frac{y^- - y^+}{|y^- - y^+|} \in \{\nu, -\nu\}$ and the left-hand side converges to μ. Thus $Y = \overline{[x]}_\mu$, and (iii) follows because the same limit is obtained for all subsequences.

To prove (ii) it is clearly enough to show that $\sup \operatorname{dist}_{[x_n]_{\mu_n}}(Y)$ converges to zero for every closed subsegment $Y \subset [x]$. Suppose this failed for some Y. Since $Y \subset S$ is closed, there is $\varepsilon > 0$ such that $B_\varepsilon(Y) \subset S$. Since $B_\varepsilon(Y)$ is convex, (iii) implies

$$d_{\mathcal{H}}\left([x_n]_{\mu_n}^{B_\varepsilon(Y)}, [x]_\mu^{B_\varepsilon(Y)}\right) \to 0. \tag{20}$$

After passing to a further subsequence (not relabelled), $\overline{[x_n]}_{\mu_n}$ converges with respect to $d_{\mathcal{H}}$ to a line segment Z. By (20) we have $Y \subset [x]_\mu^{B_\varepsilon(Y)} \subset Z$ because $[x_n]_{\mu_n}^{B_\varepsilon(Y)} \subset [x_n]_{\mu_n}$ since $B_\varepsilon(Y) \subset S$. This contradiction proves (ii).
The proof of (iv) is left to the reader.
To prove (vi) let $\varepsilon_1 > 0$ be so small that $B_{5\varepsilon_1}(\partial_i S) \cap \partial_j S = \emptyset$ for all $i \neq j$. Let $[x]_\mu$ be such that $\sup \operatorname{dist}_{\partial S}([x]_\mu) < \varepsilon_1$. Suppose that there were $i \neq j$ such that the endpoints $x^\pm$ of $[x]_\mu$ satisfy $x^- \in \partial_i S$ and $x^+ \in \partial_j S$. Then $\operatorname{dist}_{\partial_i S}(x^+) > 4\varepsilon_1$. So by convexity of $[x]_\mu = (x^- x^+)$ there is $x_0 \in [x]_\mu$ with $\operatorname{dist}_{\partial_i S}(x_0) = 2\varepsilon_1$. But then $\operatorname{dist}_{\partial_k S}(x_0) > \varepsilon_1$ also for every $k \neq i$. This would imply $\sup \operatorname{dist}_{\partial S}([x]_\mu) > \varepsilon_1$, a contradiction.

□

For $v \in \mathbb{S}^1$, $J \subset \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ Lipschitz we define

$$\operatorname{graph}_v h|_J := \{\xi v + h(\xi) v^\perp : \xi \in J\}. \tag{21}$$

A set $\Gamma \subset \mathbb{R}^2$ is called a Lipschitz graph if it is of the form (21) with J connected.

Lemma 3 *Let $S \subset \mathbb{R}^2$ be a Lipschitz domain. For all $\delta > 0$ there is $\varepsilon \in (0, \delta)$ such that the following holds: If $x_1, x_2 \in S$ and $\mu_1, \mu_2 \in \mathbb{S}^1$ are such that $[x_1]_{\mu_1} \cap [x_2]_{\mu_2} = \emptyset$, $d_{\mathcal{H}}([x_1]_{\mu_1}, [x_2]_{\mu_2}) < \varepsilon$ and $\mathcal{H}^1([x_i]_{\mu_i}) \geq \delta$ for $i = 1, 2$, then there is precisely one connected component $S([x_1]_{\mu_1}, [x_2]_{\mu_2})$ of $S \setminus ([x_1]_{\mu_1} \cup [x_2]_{\mu_2})$ which contains both $[x_1]_{\mu_1}$ and $[x_2]_{\mu_2}$ in its boundary and whose area is less than $\frac{|S|}{4}$.*
Moreover, $S([x_1]_{\mu_1}, [x_2]_{\mu_2})$ enjoys the following properties: There exist disjoint Lipschitz graphs $\Gamma_\pm \subset \partial S$ such that

$$[x_1]_{\mu_1} \cup [x_2]_{\mu_2} \cup \Gamma_+ \cup \Gamma_-$$

is a closed Jordan curve and

$$S([x_1]_{\mu_1}, [x_2]_{\mu_2}) = U_b([x_1]_{\mu_1} \cup [x_2]_{\mu_2} \cup \Gamma_+ \cup \Gamma_-).$$

In particular, $S([x_1]_{\mu_1}, [x_2]_{\mu_2})$ is simply connected. Moreover, there exists a constant $C(S)$ such that

$$\mathcal{H}^2\Big(S([x_1]_{\mu_1}, [x_2]_{\mu_2})\Big) \leq C(S) d_{\mathcal{H}}([x_1]_{\mu_1}, [x_2]_{\mu_2}) \tag{22}$$

and

$$\operatorname{diam} \Gamma_\pm \leq C(S) d_{\mathcal{H}}([x_1]_{\mu_1}, [x_2]_{\mu_2}).$$

Remarks.

(i) The hypothesis $[x_1]\cup[x_2]\subset\partial S([x_1],[x_2])$ is necessary for uniqueness: If the endpoints of the $[x_i]_{\mu_i}$ lie in the same component of ∂S then there can be two components of $S\setminus([x_1]\cup[x_2])$ with small area. See Figure 1 for an example of a set $S([x_1],[x_2])$.

(ii) The relative lower bound on $\mathcal{H}^1([x_i]_{\mu_i})$ with respect to the distance between the $[x_i]_{\mu_i}$ cannot be omitted from the hypotheses either. Otherwise it could happen that there is no disjoint pair of graphs $\Gamma_\pm\subset\partial S$ such that $x_i^*\in\Gamma_*$, and the set $S([x_1],[x_2])$ would not exist: Consider e.g. $S=B_1(0)$ and $x_1^+=x_2^+=(0,1)$, $x_1^-=(-\frac{1}{2},\frac{\sqrt{3}}{2})$ and $x_2^-=(\frac{1}{2},\frac{\sqrt{3}}{2})$. Then every subarc Γ_- of ∂S which is a graph and which contains $\{x_1^-,x_2^-\}$ must also contain $\Gamma_+=\{(0,1)\}$. Moreover, the component of $S\setminus([x_1]\cup[x_2])$ that has $[x_1]\cup[x_2]$ in its boundary has area greater than $\frac{|S|}{2}$.

Proof. We omit the indices μ_i to avoid heavy notation, and we let $x_i\in S$ and $\delta>0$ be as in the hypotheses. For $\varepsilon>0$ small enough, Lemma 2 (v) implies that there are i, j such that one endpoint of $[x_1]$ and of $[x_2]$ lies in $\partial_i S$ and the other ones in $\partial_j S$. Let us establish uniqueness of $S([x_1],[x_2])$. If $i=j$ then Lemma 16 (iv) shows that $S\setminus([x_1]\cup[x_2])$ has exactly three connected components, and at most one of them contains $[x_1]\cup[x_2]$ in its boundary. If $i\neq j$ then by Lemma 16 (ii) the set $S\setminus([x_1]\cup[x_2])$ has exactly two connected components. So the sum of their areas must be $|S|$, whence at most one can have area less than $\frac{1}{4}|S|$.
To prove existence notice that from the Lipschitz property of S and since $\sum_{*=+,-}|x_1^*-x_2^*|\leq C\varepsilon$ (by Lemma 2 (i) and with the notation used there), for small $\varepsilon>0$ there exist Lipschitz graphs $\tilde\Gamma_\pm\subset\partial S$ such that $x_1^*,x_2^*\in\tilde\Gamma_*$ $(*=+,-)$. If $x_1^*=x_2^*$ set $\Gamma_*:=\{x_1^*\}$. (Notice, however, that $x_1^+\neq x_2^+$ or $x_1^-\neq x_2^-$ since otherwise $[x_1]=[x_2]$.) If $x_1^*\neq x_2^*$ then let Γ_* be the closed subarc of $\tilde\Gamma_*$ with endpoints x_1^* and x_2^*. Clearly Γ_* is a Lipschitz graph. Thus $\operatorname{diam}\Gamma_*\leq\mathcal{H}^1(\Gamma_*)\leq C|x_1^*-x_2^*|\leq Cd_{\mathcal{H}}([x_1],[x_2])$. Hence $\Gamma_+\cap\Gamma_-=\emptyset$ if ε is small enough with respect to δ, since $|x_i^+-x_i^-|\geq\delta$ and $x_i^*\in\Gamma_*$. Since also $\Gamma_*\cap[x_i]=\emptyset$, the closed curve

$$\alpha:=\Gamma_+\cup\Gamma_-\cup[x_1]\cup[x_2]$$

is a closed Jordan curve. We define $S([x_1],[x_2])$ to be the bounded connected component $U_b(\alpha)$ of $\mathbb{R}^2\setminus\alpha$ furnished by the Jordan Curve Theorem. In the rest of this proof, for brevity we write V instead of $S([x_1],[x_2])$. We claim that

$$\mathcal{H}^2(V)\leq Cd_{\mathcal{H}}([x_1],[x_2]) \tag{23}$$

for a constant C depending only on S. As seen above, $\operatorname{diam}\Gamma_*\leq C_1d_{\mathcal{H}}([x_1],[x_2])$ for some C_1 depending only on S. With this C_1, set $R=(C_1+1)d_{\mathcal{H}}([x_1],[x_2])$. Then (23) follows if we show that $V\subset B_R([x_1])$, since $\mathcal{H}^1([x_1])\leq\operatorname{diam}S$.
Since $\operatorname{diam}\Gamma_\pm\leq\frac{C_1}{C_1+1}R$, we have $\Gamma_\pm\subset B_R([x_1])$, because both Γ_+ and Γ_- intersect the closure of $[x_1]$. By definition of $d_{\mathcal{H}}$ we also have $[x_2]\subset B_R([x_1])$. So $\alpha\subset B_R([x_1])$. Thus $B_R([x_1])$ contains $V=U_b(\alpha)$ by Remark 8, and (23) is proven.
Next we claim that V is contained in S. In fact, since $\alpha\subset\overline{U_b(\partial_0S)}$, Remark 8 implies that $V\subset U_b(\partial_0S)$. We claim that also $V\subset U_\infty(\partial_mS)$ for all $m=1,2,\ldots$. In fact, otherwise by openness of V there would be $m\geq1$ with $U_b(\partial_mS)\cap V\neq\emptyset$. But $\partial V\subset\bar S$ does not intersect $U_b(\partial_mS)$. Hence by connectedness $U_b(\partial_mS)\subset V$. For ε small enough this contradicts (23). By (18), we conclude that indeed $V\subset S$.
Finally, we claim that V is a connected component of $S\setminus([x_1]\cup[x_2])$. In fact, since V is an open subset of S, there is at least one connected component S' of $S\setminus([x_1]\cup[x_2])$ with $S'\cap V\neq\emptyset$. But $S'\cap\partial V\subset S'\cap(\partial S\cup[x_1]\cup[x_2])=\emptyset$. Thus by connectedness $S'\subset V$. Also, since $V\subset S$ we have $V\cap\partial S'=V\cap(S\cap\partial S')=V\cap([x_1]\cup[x_2])=\emptyset$. So by connectedness V is contained in S'. $\square$

3 Countably developable mappings and geometry of the set of local constancy

We will now define countably developable mappings, which are the object of study in this paper. They almost agree with the class of mappings satisfying condition (L) from [15]. The only difference is that we require, in addition, that they be continuous (in the interior). Some of the theory developed here also carries over to discontinuous mappings as found in [11]. We focus on continuous mappings because $W^{2,2}$ isometric immersions are C^1. As explained in the introduction, since we do not make any integrability assumption, our results also apply to the setting in [20].

3.1 Countably developable mappings

Let $S \subset \mathbb{R}^2$ be a bounded domain, let $X \subset S$ and let $P \in \mathbb{N}$. A mapping $q : X \to \mathbb{S}^1$ is called an S-ruling (on X) if

$$[x]^S_{q(x)} \cap [y]^S_{q(y)} \neq \emptyset \implies [x]^S_{q(x)} = [y]^S_{q(y)} \tag{24}$$

whenever $x, y \in X$. A mapping

$$f : X \to \mathbb{R}^P$$

is called S-*developable* on X if there exists an S-ruling for f, i.e. an S-ruling $q_f : X \to \mathbb{S}^1$ such that

$$f \text{ is constant on } X \cap [x]^S_{q_f(x)} \text{ for all } x \in X.$$

Note that f being S-developable on $X_1 \subset S$ and on $X_2 \subset S$ does not imply that f is S-developable on $X_1 \cup X_2$.

Definition 1 *A mapping $f \in C^0(S; \mathbb{R}^P)$ is called countably S-developable if f is S-developable on $S \setminus C_f$. Here, C_f is as in (2).*

We will see later that, if f is countably developable, then $[x] \subset S \setminus C_f$ for all $x \in S \setminus C_f$.

If no confusion can arise, we will write $[x]$ instead of $[x]^S_{q_f(x)}$, and we will omit the prefix '$S-$' in the above definitions. Notice that, in general, a countably developable mapping f consists of infinitely many developable 'pieces'. One key result of this paper is that every countably developable mapping can be approximated in a strong sense by countably developable mappings which consist of only finitely many developable pieces, cf. Theorem 3 below.

Remark 1 *If S is a bounded domain, $X \subset S$ and q is an S-ruling on X, then q is locally Lipschitz on X if regarded as a mapping into the projective space $\mathbb{P}^1$.*

Proof. Continuity of q follows from Lemma 2 (iv). As observed in [14], proof of Proposition 2.30, the condition (24) in fact implies that q is locally Lipschitz near $x \in X$ with a Lipschitz constant of the order $\text{dist}_{\partial S}(x)$. □

Observe that the continuity of the ruling remains true if f is discontinuous. The following lemma proves uniqueness (on $S \setminus C_f$) of the ruling q_f associated with a countably developable mapping f.

Lemma 4 *If $f \in C^0(S; \mathbb{R}^P)$ is countably developable, then the S-ruling $q_f : S \setminus C_f \to \mathbb{S}^1$ is continuous and unique if regarded as a mapping into the projective space $\mathbb{P}^1$.*

Proof. To prove uniqueness, let

$$q_f^{(1)}, q_f^{(2)} : S \setminus C_f \to \mathbb{S}^1$$

be two S-rulings for f. Suppose there were $x \in S \setminus C_f$ such that $q_f^{(1)}(x) \neq q_f^{(2)}(x)$ (in $\mathbb{P}^1$). Then

$$[x]^S_{q_f^{(1)}(x)} \cap [x]^S_{q_f^{(2)}(x)} = \{x\}$$

and

$$f = f(x) \text{ on } [x]^S_{q_f^{(1)}(x)} \cup [x]^S_{q_f^{(2)}(x)}.$$

But $q_f^{(1)}$ is continuous. So there is $\delta > 0$ such that $[y]^S_{q_f^{(1)}(y)}$ intersects $[x]^S_{q_f^{(2)}(x)}$ for all $y \in B_\delta(x) \setminus C_f$. Thus $f(y) = f(x)$ for all $y \in B_\delta(x) \setminus C_f$. Hence by definition of C_f and by continuity of f we have $f \equiv f(x)$ on $B_\delta(x)$. This contradicts the fact that $x \notin C_f$. □

3.2 Geometry of the set of local constancy C_f

In [8] it is shown that each connected component U of C_f is a convex polygon when the domain S is convex, and a related fact if S is simply connected, cf. also [14]. The situation for domains which are not simply connected is more delicate, mainly due to a topological difference: A multiply connected domain is, in general, not disconnected by a single curve $\gamma \subset S$ with endpoints on the boundary ∂S, i.e., the set $S \setminus \gamma$ can still be a connected set. This will be the case when the endpoints of the curve lie on different connected components of the boundary.

If $S \subset \mathbb{R}^2$ is a continuous domain, $X \subset S$ and $f : X \to \mathbb{R}^P$ is S-developable, and if $\Gamma_-, \Gamma_+ \subset \partial S$ are connected, we define

$$A^{f,X}(\Gamma_-, \Gamma_+) := \left\{ x \in X : [x]^S_{q_f(x)} = (x^-, x^+) \text{ with } x^- \in \Gamma_- \text{ and } x^+ \in \Gamma_+ \right\}. \tag{25}$$

For $i, j \in \{0, ..., N_S\}$ we also set $A^{f,X}_{ij} := A^f(\partial_i S, \partial_j S)$, and we will omit the indices f, X if they are clear from the context. For instance, if f is countably developable, then it is understood that $X = S \setminus C_f$ (later that $X = S \setminus \hat{C}_f$).

Definition 2 *If $S \subset \mathbb{R}^2$ is a bounded domain, then a subset U of S is said to satisfy condition (B_f) for an S-developable mapping $f : S \cap \partial U \to \mathbb{R}^P$ provided that U is open, connected and that $[x]^S_{q_f(x)} \subset \partial U$ for all $x \in S \cap \partial U$.*

Remark 2 *(i) The condition (B_f) will typically appear in the following context: The mapping $f \in C^0(S; \mathbb{R}^P)$ is countably developable, and $U \subset S$ is such that $C_f \cap \partial U = \emptyset$. We will later introduce the subset $\hat{C}_f$ of C_f and show that f is developable on $S \setminus \hat{C}_f$, cf. Proposition 9. Then condition (B_f) also makes sense if $\hat{C}_f \cap \partial U = \emptyset$.*

(ii) Apart from the trivial example $U = S \setminus [x]$, in what follows we will encounter mainly three kinds of sets satisfying condition (B_f): The connected components of C_f, the set S^f_δ defined below and the sets $[\Gamma(0, T)]$ defined below.

The following lemma is of basic importance in the rest of this paper.

Lemma 5 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $U \subset S$ satisfy condition (B_f) for some S-developable mapping $f : S \cap \partial U \to \mathbb{R}^P$. Then U has finite perimeter. Moreover, there exists a countable subset Z_U of $S \cap \partial U$ which satisfies (i) and (ii) below. Moreover, (iii) through (viii) hold:*

(i) *If* $x, y \in Z_U$ *with* $x \neq y$ *then* $[x] \cap [y] = \emptyset$.

(ii) $S \cap \partial U = \bigcup_{x \in Z_U} [x]$.

(iii) *If* $\Gamma_-, \Gamma_+ \subset \partial S$ *are disjoint and connected then* $\# \ Z_U \cap A(\Gamma_-, \Gamma_+) \leq 2$. *In particular,* $\# \ Z_U \cap A_{ij} \leq 2$ *whenever* $i, j \in \{0, ..., N_S\}$ *with* $i \neq j$.

(iv) $\sum_{x \in Z_U \cap A_{ii}} \mathcal{H}^1([x]) \leq \mathcal{H}^1(\partial_i S)$ *for all* $i \in \{0, ..., N_S\}$.

(v) *If* $x \in A_{ii} \cap \partial U$ *for some* $i \in \{0, ..., N_S\}$*, then* $S \setminus [x]$ *consists of two components* S_x^1 *and* S_x^2*, and* $U \subset S_x^1$. *If* $x, y \in Z_U$ *with* $x \neq y$ *then*

$$\overline{S}_x^2 \cap \overline{S}_y^2 \cap S = \emptyset; \tag{26}$$

in particular, $S_x^2 \cap S_y^2 = \emptyset$.

(vi) *For all* $x \in S \cap \partial U$ *there is* $r > 0$ *such that* $B_r(x) \subset S$ *and* $B_r(x) \cap \partial U = B_r(x) \cap [x]$. *Moreover,* $B_r(x) \setminus [x]$ *consists of two open half-disks* B_x^1 *and* B_x^2*, and* $B_x^1 \subset U$. *The half-disk* B_x^2 *is contained either in* $S \setminus \bar{U}$ *or in* U. *If* $x \in A_{ii}$ *for some* $i \in \{0, ..., N_S\}$ *then* $B_x^2 \subset S_x^2 \subset S \setminus \bar{U}$.

(vii) *If* $S \cap \partial U \subset \bigcup_{i=0}^{N_S} A_{ii}$ *then*

$$S \setminus \bigcup_{x \in Z_U} [x] = U \cup \bigcup_{x \in Z_U} S_x^2. \tag{27}$$

(viii) *Let* $V \subset S$ *satisfy* (B_f). *If* $U \cap V = \emptyset$ *and* $\bar{U} \cap \bar{V} \neq \emptyset$ *then* $U \cup V \cup [x]$ *satisfies* (B_f) *for all* $x \in S \cap \bar{U} \cap \bar{V} = S \cap \partial U \cap \partial V$. *Moreover,* $\mathrm{int}(\bar{U} \cup \bar{V})$ *satisfies* (B_f)*, and*

$$\mathrm{int}(\bar{U} \cup \bar{V}) = U \cup V \cup (S \cap \partial U \cap \partial V). \tag{28}$$

Remarks.

(i) In statement (iii) one can have $\# Z_U \cap A_{ij} = 1$: Consider e.g. $S = B_2(0) \setminus \bar{B}_1(0)$ and let $[x]$ be the open segment with endpoints $(0, 1)$ and $(0, 2)$. Then $U := S \setminus [x]$ satisfies (B_f) (e.g. for $f \equiv e_2$)

(ii) If $x \in A_{ij}^f$ with $i \neq j$ in statement (vi) then one can have $B_r(x) \setminus [x] \subset U$. See the previous example.

Proof. First of all notice that (iii) and (iv) imply that $\sum_{x \in Z_U} \mathcal{H}^1([x]) < \infty$, so U has finite perimeter by (ii).

Claim #1. If Γ_+, $\Gamma_- \subset \partial S$ are disjoint and connected, then there are points $x_1, x_2 \in S \cap \partial U$ such that $A(\Gamma_-, \Gamma_+) \cap \partial U \subset [x_1] \cup [x_2]$ (possibly $[x_1] = [x_2]$). In particular, if $i \neq j$ then $\partial U \cap A_{ij} \subset [x_1] \cup [x_2]$ for some $x_1, x_2 \in S \cap \partial U$.

To prove Claim #1 let $x_1, x_2, x_3 \in \partial U \cap A(\Gamma_-, \Gamma_+)$. Then by the hypothesis on ∂U we have $[x_k] \subset \partial U \cap S$ for $k = 1, 2, 3$. Suppose for contradiction that the $[x_k]$ were pairwise disjoint. Set $V_m = \mathcal{C}(S \setminus ([x_k] \cup [x_l]); x_m)$, where (k, l, m) is a permutation of $(1, 2, 3)$, so $m = 1, 2, 3$. Then V_m is open and contains $x_m \in \partial U$, so V_m intersects U. But $\partial V_m \subset \partial S \cup [x_1] \cup [x_2] \cup [x_3] \subset \partial S \cup \partial U$. So U does not intersect ∂V_m. Thus $U \subset V_m$ by connectedness. This is true for $m = 1, 2, 3$. Applying Lemma 17 with $\gamma_k = [x_k]$, we conclude that $V_1 \cap V_2 \cap V_3$ is empty. This contradiction proves Claim #1.

Now let $k \in \{0, ..., N_S\}$ and let $x \in A_{kk} \cap \partial U$. Denote by $x^{\pm}$ the endpoints of $[x]$ and by $\partial_k^1 S$ and $\partial_k^2 S$ the connected components of $\partial_k S \setminus \{x^-, x^+\}$. By Lemma

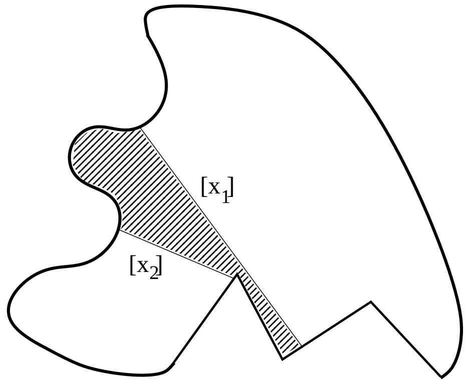

Figure 1: An example for $S([x_1]_{\mu_1}, [x_2]_{\mu_2})$. Notice that it is not contained in a cone with boundaries $[x_i]^{\mathbb{R}}_{\mu_i}$.

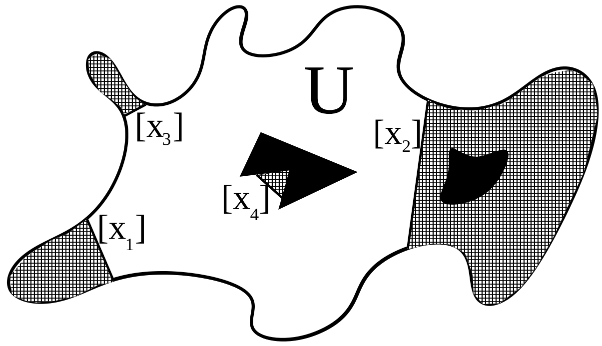

Figure 2: The sets S^2_x: The white region is U satisfying (B_f). The regions with a pattern are the $S^2_{x_i}$ for $x_i \in Z_U$, where $i = 1, ..., 4$. The two black holes belong to the complement of S.

16 (i), the set $S \setminus [x]$ consists of exactly two connected components S^1_x and S^2_x, and $\{0, ..., N_S\} \setminus \{k\}$ can be partitioned into two subsets I_1 and I_2 such that

$$\partial S^i_x = \partial^i_k S \cup \overline{[x]} \cup \bigcup_{j \in I_i} \partial_j S \text{ for } i = 1, 2. \tag{29}$$

By connectedness, U is contained in one of the two components of $S \setminus [x]$, and we choose labels such that $U \subset S^1_x$. So U does not intersect the closure of S^2_x. We define $J_x := \partial^2_k S$.

Claim #2. If $x, y \in A_{kk} \cap \partial U$ with $[x] \cap [y] = \emptyset$ then $S^2_x \cap S^2_y = \emptyset$ and $J_x \cap J_y = \emptyset$. In fact, if x, y are as above then, since $U \subset S^1_x$, we have $y \in S \cap \bar{S}^1_x = [x] \cup S^1_x$ by (29). Since $y \notin [x]$ this implies $y \in S^1_x$, and since $[y]$ does not intersect ∂S^1_x and is connected, we conclude that $[y] \subset S^1_x$.

Thus $S^2_x \cap \partial S^2_y = S^2_x \cap [y]$ is empty. Hence by connectedness of S^2_x either $S^2_x \subset S^2_y$ or $S^2_x \cap S^2_y = \emptyset$. The analogous argument with the roles of x and y interchanged then implies that either $S^2_x \cap S^2_y = \emptyset$ or $S^2_y = S^2_x$. But the latter case is impossible because $[x] \neq [y]$. So S^2_x and S^2_y are disjoint. By openness of the S^i_x, this implies that $S^2_x \subset S^1_y$ and $S^2_y \subset S^1_x$. Hence $\bar{S}^2_x \cap \bar{S}^2_y \subset \bar{S}^1_y \cap \bar{S}^2_y \subset \overline{[y]}$ by (29) (with y instead of x). Swapping the roles of x and y we conclude that

$$\bar{S}^2_x \cap \bar{S}^2_y \subset \overline{[x]} \cap \overline{[y]} \subset \{x^+, x^-\} \cap \{y^+, y^-\}. \tag{30}$$

Since $J_x \subset \bar{S}^2_x \setminus \{x^+, x^-\}$ and $J_y \subset \bar{S}^2_y \setminus \{y^+, y^-\}$, we deduce from (30) that J_x and J_y are disjoint. This proves the claim.

Since J_x is an arc with the same endpoints as $[x]$, we have

$$\mathcal{H}^1([x]) \leq \mathcal{H}^1(J_x). \tag{31}$$

By applying Zorn's Lemma to the family of subsets $F \subset A_{ii} \cap \partial U$ which satisfy $[x] \neq [y]$ whenever $x, y \in F$ and $x \neq y$, we obtain a set Z'_i with the following property: $Z'_i \subset A_{ii} \cap \partial U$ and $\partial U \cap A_{ii} = \bigcup_{x \in Z'_i} [x]$, and $[x] \neq [y]$ whenever $x, y \in Z'_i$ with $x \neq y$. For every countable subset $Z''_i \subset Z'_i$, by (31) and the disjointness of the J_x (see Claim #2) we have $\mathcal{H}^1(\bigcup_{x \in Z''_i} [x]) = \sum_{x \in Z''_i} \mathcal{H}^1([x]) \leq \sum_{x \in Z''_i} \mathcal{H}^1(J_x) \leq \mathcal{H}^1(\partial_i S)$, which is finite. So Z'_i is countable with

$$\mathcal{H}^1(\bigcup_{x \in Z'_i} [x]) \leq \mathcal{H}^1(\partial_i S). \tag{32}$$

By Claim #1, whenever $i, j \in \{0, ..., N_S\}$ with $i \neq j$ and $A_{ij} \cap \partial U \neq \emptyset$, there exist $x_{ij}^{(1)}, x_{ij}^{(2)}$ (possibly equal) such that

$$A_{ij} \cap \partial U = [x_{ij}^{(1)}] \cup [x_{ij}^{(2)}].$$

Set

$$Z_U := \bigcup_{i=0}^{N_S} Z_i' \cup \bigcup_{\{i<j : A_{ij} \cap \partial U \neq \emptyset\}} \{x_{ij}^{(1)}, x_{ij}^{(2)}\}.$$

This set clearly satisfies conditions (i) and (ii). It satisfies (iii) by definition, it satisfies (iv) by (32), and it satisfies (v) by Claim #2. Observe that (26) follows from (30).

To prove (vi) let $x \in S \cap \partial U$ and let R be such that $B_{2R}(x) \subset S$. Since by (iii) and (iv) the sum $\sum_{x \in Z_U} \mathcal{H}^1([x])$ converges, the set $Z' := \{y \in Z_U : [y] \cap B_R(x) \neq \emptyset\} \setminus \{x\}$ is finite, because if $y \in Z'$ then $\mathcal{H}^1([y]) \geq R$ (in fact, the endpoints of $[y]$ lie outside $B_{2R}(x)$ since they are outside S). By definition of Z' we have

$$B_R(x) \cap \partial U = B_R(x) \cap S \cap \partial U \subset [x] \cup \bigcup_{y \in Z'} [y]. \tag{33}$$

Moreover, for $y \in Z'$, since $B_{2R}(x) \subset S$, by maximality of $[x]$ and $[y]$ we have $\overline{[x]} \cap \bar{B}_R(x) \subset [x]$ and $\overline{[y]} \cap \bar{B}_R(x) \subset [y]$. So these two compact sets do not intersect for any $y \in Z'$, so they have positive distance for each $y \in Z'$. Since Z' is finite, from (33) we deduce that there is $r \in (0, R)$ with $B_r(x) \cap \partial U \subset [x]$, as claimed.
The endpoints of $[x]$ lie outside $B_r(x)$, so $B_r(x) \setminus [x]$ consists of two components (see e.g. Lemma 16), which of course are open half-disks B_x^1 and B_x^2. But none of them intersects ∂U, so by connectedness each one is either contained in U or in $S \setminus \bar{U}$. Since $x \in \partial U$ we have $B_r(x) \cap U \neq \emptyset$. Hence $(B_x^1 \cup B_x^2) \cap U \neq \emptyset$ because $[x] \cap U = \emptyset$. We choose the labels such that $B_x^1 \cap U \neq \emptyset$, so $B_x^1 \subset U$. If $x \in A_{ii}$ for some i, that is, if both endpoints of $[x]$ lie in the same connected component of ∂S, then by part (v), the set $S \setminus [x]$ consists of two components S_x^1 and S_x^2, and $U \subset S_x^1$. By Lemma 16 (i) we have $[x] \subset \partial S_x^1 \cap \partial S_x^2$. So $B_r(x)$ intersects S_x^2. Since $B_x^1 \subset U \subset S_x^1$, this implies $B_x^2 \cap S_x^2 \neq \emptyset$. Hence by connectedness $B_x^2 \subset S_x^2$.

Let us prove (vii). Since $U \subset S$ is open and since by (ii) we have $\bigcup_{x \in Z_U} [x] \subset \partial U$, we have $U \subset S \setminus \bigcup_{x \in Z_U} [x]$. Let $x, y \in Z_U$ with $x \neq y$. By (30) and since $\overline{[y]} \subset \bar{S}_y^2$ (see e.g. Lemma 16 (i)), we have $\bar{S}_x^2 \cap \overline{[y]} \subset \partial S$, so $S_x^2 \cap [y] = \emptyset$. Since this is true for all $y \in Z_U \setminus \{x\}$ and by openness also $S_x^2 \cap [x] = \emptyset$, we conclude that $S_x^2 \subset S \setminus \bigcup_{z \in Z_U} [z]$. This proves one inclusion.
To prove that $S \setminus \bigcup_{x \in Z_U} [x] \subset U \cup \bigcup_{x \in Z_U} S_x^2$, let V be a connected component of $S \setminus \bigcup_{x \in Z_U} [x]$. By part (ii) we have $V \cap \partial U = \emptyset$, so if $V \cap U \neq \emptyset$ then $V \subset U$ by connectedness (in fact, $V = U$ by maximality of V). It remains to consider the case $V \cap U = \emptyset$. By definition, $S \cap \partial V \subset \bigcup_{x \in Z_U} [x]$. And $S \cap \partial V$ is nonempty because otherwise $V = S$, so it would intersect U. On the other hand, $\bigcup_{x \in Z_U} [x] \subset S \cap \partial U$ by (ii). So there is $x \in S \cap \partial U \cap \partial V$. By definition, $U \subset S_x^1$. By connectedness either $V \subset S_x^1$ or $V \subset S_x^2$. By (vi), for r small enough, $B_r(x) \setminus [x]$ consists of two half-disks B_x^1, B_x^2, with $B_x^1 \subset U \subset S_x^1$ and $B_x^2 \subset S_x^2$. Since $V \cap U = \emptyset$ and $x \in \partial V$, this implies that V intersects S_x^2. We conclude that $V \subset S_x^2$.

To prove (viii) first notice that by disjointness we have $\bar{U} \cap \bar{V} = \partial U \cap \partial V$. Let $x \in S \cap \partial U \cap \partial V$ and let $z \in [x]$. Notice that $[x] \subset S \cap \partial U \cap \partial V$ because U and V satisfy (B_f). Hence we can apply (vi) to z to find $r > 0$ such that $B_r(z) \setminus [x]$ (notice that $[x] = [z]$) consists of two half-disks $B_z^1 \subset U$ and $B_z^2 \subset S \setminus \bar{U}$. Indeed, if both half-disks were contained in U then $B_r(z) \cap V = \emptyset$, contradicting the fact that $z \in \partial V$. Repeating this argument with the roles of U and V interchanged shows

that $B_z^2 \subset V$. Hence $B_r(z) \subset U \cup V \cup [x]$. Since $z \in [x]$ was arbitrary and U, V are open by hypothesis, this proves that $U \cup V \cup [x]$ is open. It also implies that $U \cup V \cup [x]$ is connected, since so are U, V and $[x]$, and all three intersect $B_r(z)$.
Next notice that

$$\partial(U \cup V \cup [x]) = (\partial U \cup \partial V) \setminus [x] \tag{34}$$

(Indeed, by openness of $U \cup V \cup [x]$ and since $[x] \subset \bar{U}$, we have $\partial(U \cup V \cup [x]) = (\bar{U} \cup \bar{V}) \setminus (U \cup V \cup [x]) = \bar{U} \setminus (U \cup [x]) \cup \bar{V} \setminus (V \cup [x])$ because $\bar{U} \cap V = \bar{V} \cap U = \emptyset$. This proves (34).) From (34) one immediately deduces that $y \in S \cap \partial(U \cup V \cup [x])$ implies that $[y] \subset \partial(U \cup V \cup [x])$. This proves that $U \cup V \cup [x]$ satisfies (B_f).
Applying the above argument to all $x \in S \cap \partial U \cap \partial V$ proves that $U \cup V \cup (S \cap \partial U \cap \partial V)$ is open. By Theorem IV.1.2 in [17] it is also connected, since for an arbitrary $x \in S \cap \partial U \cap \partial V$ we have

$$U \cup V \cup [x] \subset U \cup V \cup (S \cap \partial U \cap \partial V) \subset \bar{U} \cup \bar{V} = \text{closure of } (U \cup V \cup [x]),$$

and we know that $U \cup V \cup [x]$ is connected.
Since $U \cup V \cup (S \cap \partial U \cap \partial V)$ is open, it is clearly contained in $\operatorname{int}(\bar{U} \cup \bar{V})$. The converse inclusion follows from $\operatorname{int}(\bar{U} \cup \bar{V}) \subset \operatorname{int} \bar{S} = S$ (because S is a continuous domain) and from $\operatorname{int}(\bar{U} \cup \bar{V}) \subset U \cup V \cup (\partial U \cap \partial V)$. (The latter holds for all disjoint open sets U and V.) Thus (28) is proven. In particular, $\operatorname{int}(\bar{U} \cup \bar{V})$ is connected. Moreover, (28) implies $\partial \operatorname{int}(\bar{U} \cup \bar{V}) = \bar{U} \cup \bar{V} \setminus \Big(U \cup V \cup (S \cap \partial U \cap \partial V)\Big)$. Hence

$$S \cap \partial \operatorname{int}(\bar{U} \cup \bar{V}) = \Big(S \cap \partial U \setminus \partial V\Big) \cup \Big(S \cap \partial V \setminus \partial U\Big). \tag{35}$$

Thus if $y \in S \cap \partial \operatorname{int}(\bar{U} \cup \bar{V})$ then (after possibly swapping U and V) we have $y \in (S \cap \partial U) \setminus \partial V$. Hence $[y] \subset \partial U$ because U satisfies (B_f). On the other hand, if $[y] \cap \partial V \neq \emptyset$ then $[y] \subset \partial V$ because V satisfies (B_f). But this would imply $y \in \partial V$, a contradiction. We conclude that $[y] \subset S \cap \partial U \setminus \partial V \subset \partial \operatorname{int}(\bar{U} \cup \bar{V})$ by (35). Therefore, $\operatorname{int}(\bar{U} \cup \bar{V})$ satisfies (B_f). □

Proposition 7 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S; \mathbb{R}^P)$ be countably developable. Then C_f consists of countably many connected components. Every connected component U of C_f satisfies condition (B_f). More precisely, $S \cap \partial U \subset S \setminus C_f$, and for all $x \in S \cap \partial U$ there is $R > 0$ such that for all $r \in (0, R)$, one component of $B_r(x) \setminus [x]$ is contained in U and the other one in $S \setminus \bar{U}$.*
Moreover, if U_1, U_2 are connected components of C_f with $\bar{U}_1 \cap \bar{U}_2 \cap S \neq \emptyset$ then $U_1 = U_2$.

Proof. Let U be a connected component of C_f. So U is open since so is C_f. Hence there can be only countably many such U. Clearly $\partial U \cap C_f = \emptyset$.

Claim #1. If $x \in S \setminus C_f$ then $[x] \cap C_f = \emptyset$.
In fact, let $x \in S \setminus C_f$, so $[x]$ is well defined, and suppose for contradiction that there were $y \in [x] \cap C_f$. For all $\delta > 0$ there exists $x_\delta \in E_\delta := B_\delta(x) \cap (S \setminus C_f)$ such that $f(x_\delta) \neq f(x)$. In fact, otherwise $f(z) = f(x)$ for all $z \in E_\delta$, but since f is constant on each connected component of $B_\delta(x) \setminus E_\delta$, continuity of f would imply $f|_{B_\delta(x)} = f(x)$, contradicting the fact that $x \notin C_f$.
Since $[xy] \subset S$ is compact, there is $\delta_0 > 0$ such that $S' := B_{\delta_0}([xy]) \subset S$. Since $x_\delta \to x$ as $\delta \downarrow 0$, Lemma 2 (iv) implies that $d_{\mathcal{H}}([x_\delta]^{S'}, [x]^{S'}) \to 0$. So for small δ we have $[x_\delta]^{S'} \cap B_\delta(y) \neq \emptyset$ because $y \in [x]^{S'}$. But for $\delta > 0$ small enough we have $B_\delta(y) \subset C_f$. Since $f|_{B_\delta(y)}$ and $f|_{[x_\delta]}$ are constant, we deduce that $f(x_\delta) = f(y) = f(x)$ for all δ small enough, a contradiction.

Claim #2. If $x \in S \cap \partial U$ then there is $r > 0$ such that $\bar{B}_r(x) \subset S$ and one connected component of $B_r(x) \setminus [x]$ is contained in U and the other one in $S \setminus \bar{U}$. In particular, $[x] \subset S \cap \partial U$.

To prove this, let $x \in \partial U \cap S$, so $x \in S \setminus C_f$, $[x]$ is well defined and $[x] \cap C_f = \emptyset$ by Claim #1. We choose coordinates such that x is the origin and $[x] \subset \mathbb{R} \times \{0\}$. (This fixes the coordinates up to a 180 degree rotation.) Let $R > 0$ be such that $\bar{B}_{2R}(x) \subset S$. So the endpoints of $[x]$ lie in $\mathbb{R}^2 \setminus B_R(x)$, so $B_R(x) \setminus [x] = B_R(0) \setminus (\mathbb{R} \times \{0\})$ consists of exactly two connected components (see e.g. Lemma 16 (i)), namely the half disks

$$B_R^+ := \{(z_1, z_2) \in B_R(0) : z_2 > 0\} \text{ and } B_R^- := \{(z_1, z_2) \in B_R(0) : z_2 < 0\}. \tag{36}$$

We claim that

$$x \in \overline{B_R(x) \setminus (C_f \cup [x])}. \tag{37}$$

In fact, otherwise there would exist $r \in (0, R)$ such that $B_r(x)$ does not intersect $S \setminus (C_f \cup [x])$. Since $B_R(x) \subset S$ this implies that $B_r(x) \setminus [x]$ is contained in C_f. By continuity of f this would imply $B_r(x) \subset C_f$, whence $x \in C_f$, a contradiction proving (37).

Claim #3. Let $* \in \{-, +\}$. If x is contained in the closure of $B_R^* \setminus C_f$ then $B_r^* \cap \bar{U} = \emptyset$ for some $r \in (0, R)$. (Here B_r^* is defined in analogy to B_R^*.)

Assume we had shown Claim #3. After possibly rotating coordinates by 180 degree, (37) implies that x is contained in the closure of $B_R^+ \setminus C_f$. So by Claim #3 we have

$$B_r^+ \cap \bar{U} = \emptyset \text{ for some } r > 0. \tag{38}$$

But $B_r(x) \setminus [x]$ intersects U since $x \in \partial U$ and $[x] \cap U = \emptyset$. So by (38) we must have $B_r^- \cap U \neq \emptyset$. So Claim #3 implies that x is not contained in the closure of $B_R^- \setminus C_f$, i.e. there is $r \in (0, R)$ (possibly smaller than the one above, but after shrinking one we may of course assume that they are equal) such that $B_r^- \setminus C_f = B_r(x) \cap (B_R^- \setminus C_f)$ is empty. That is, $B_r^- \subset C_f$. Since B_r^- is connected and intersects U, by maximality of U we conclude $B_r^- \subset U$. This together with (38) proves the first part of Claim #2. In particular, $B_r(x) \cap [x] \subset S \cap \partial U$. It remains to prove that $[x] \subset \partial U$. But by the first part, the set $[x] \cap \partial U$ is relatively open in $[x]$: If $z \in [x] \cap \partial U$ then $B_r(z) \cap [x] = B_r(z) \cap [z] \subset \partial U$ as well. Since ∂U is closed, $[x] \cap \partial U$ is also relatively closed in $[x]$. So $[x] \cap \partial U = [x]$ since $[x]$ is connected. This proves Claim #2.

It remains to prove Claim #3. After possibly rotating the coordinates by 180 degree, we may assume that $* = +$, i.e. there are $y_n \in B_R^+ \setminus C_f$ with $y_n \to x$. After an appropriate choice of sign for $q_f(y_n)$, Lemma 4 implies $q_f(y_n) \to q_f(x) = e_1$. Hence $q_f(y_n) \cdot e_1 \neq 0$ for large n, so we can label the endpoints $y_n^\pm$ of $[y_n]$ such that $y_n^- \cdot e_1 < y_n^+ \cdot e_1$, and with each y_n we can associate the unique affine function $f_n : \mathbb{R} \to \mathbb{R}$ with graph $f_n = [y_n]^{\mathbb{R}^2}$. Since $y_n \to x$ and $q_f(y_n) \to e_1$, affinity of the f_n implies that

$$f_n \to 0 \text{ uniformly on compact intervals.} \tag{39}$$

Set $Q = [-R, R]^2$. Since $Q \subset B_{2R}(0) \subset S$ and $y_n \in Q$ for large n (recall that x is the origin), we have $[y_n]^{\mathbb{R}^2} \cap Q = [y_n] \cap Q$. By (39) we have $\operatorname{graph} f_n \cap Q = \operatorname{graph} f_n|_{[-R,R]}$ for large n. We conclude that

$$\operatorname{graph} f_n|_{[-R,R]} = [y_n] \cap Q \text{ for } n \text{ large enough.} \tag{40}$$

Using this we have $\operatorname{graph} f_n|_{[-R,R]} \cap (\mathbb{R} \times \{0\}) = [y_n] \cap [x] \cap Q$, which is empty. Since $f_n(y_n \cdot e_1) = y_n \cdot e_2 > 0$, we deduce $f_n > 0$ on $[-R, R]$. Similarly, $f_n \neq f_m$ on $[-R, R]$ for $n \neq m$ since then $[y_n] \cap [y_m] = \emptyset$. Thus by (39), after passing to subsequences we have

$$0 < f_{n+1} < f_n \text{ on } [-R, R] \text{ for large } n. \tag{41}$$

By Lemma 2 (ii) we have $\mathcal{H}^1([y_n]) > \frac{1}{2}\mathcal{H}^1([x])$ for large n, and so we can apply Lemma 3 (with $\delta = \frac{1}{2}\mathcal{H}^1([x])$, but δ will denote something else below) to obtain ε

with the properties stated there. After passing to a further subsequence we may assume that

$$d_{\mathcal{H}}([y_n],[y_m]) < \varepsilon \text{ for large } n,m. \tag{42}$$

Now let N so large that (41) and (42) hold for $n, m \geq N$.
By (41) we have $\delta := \min f_N([-R,R]) > 0$. For $n > N$ set $S_n = \Big\{z \in \mathbb{R}^2 : z\cdot e_1 \in [-R,R], z\cdot e_2 \in \big(f_n(z\cdot e_1), f_N(z\cdot e_1)\big)\Big\}$. By (41), the S_n form an increasing sequence. Moreover,

$$\Big([-R,R]\times(0,\delta)\Big)\cap C_f \subset \bigcup_{n\geq N} S_n. \tag{43}$$

In fact, let $z \in S\setminus\bigcup_{n\geq N}[y_n]$ be such that $|z\cdot e_1| \leq R$ and $z\cdot e_2 \in (0,\delta)$. Then $z\cdot e_2 < \delta \leq f_N(z\cdot e_1)$. And by (39) we have $z\cdot e_2 > f_n(z\cdot e_1)$ for large n. This proves (43) since $C_f \subset S\setminus[y_n]$ for all n (recall that $[y_n]\cap C_f = \emptyset$ by Claim #1).
We claim that U does not intersect $[-R,R]\times(0,\delta)$. To prove this, let us assume the contrary, so by (43) and since $U \subset C_f$ we have that U intersects S_n for all n large enough. We claim that this implies

$$U \subset S([y_n],[y_N]) =: V_n \text{ for all } n \text{ large enough}. \tag{44}$$

Here $S([y_n],[y_N]) = V_n$ is the connected component of $S\setminus([y_n]\cup[y_N])$ furnished by Lemma 3 (the hypotheses of this lemma are satisfied by (42)). In fact, U is connected and it does not intersect ∂V_n because $\partial V_n \subset \partial S\cup[y_n]\cup[y_N]$ (see Lemma 3). So we must show that S_n is contained in V_n, since then U intersects V_n and therefore $U \subset V_n$. But S_n is connected and it does not intersect ∂V_n, so it is enough to prove that $S_n \cap V_n \neq \emptyset$.
By Lemma 3 there are arcs $\Gamma_\pm^{(n)} \subset \partial S$ such that $\partial V_n = \alpha_n := [y_n]\cup[y_N]\cup\Gamma_+^{(n)}\cup\Gamma_-^{(n)}$. And $\varepsilon_n := \operatorname{diam}\Gamma_+^{(n)} + \operatorname{diam}\Gamma_-^{(n)} \leq C d_{\mathcal{H}}([y_n],[y_N])$ and one endpoint of $\Gamma_\pm^{(n)}$ is $y_n^\pm$. So $\Gamma_+^{(n)} \subset B_{\varepsilon_n}(y_n^+)$ and $\Gamma_-^{(n)} \subset B_{\varepsilon_n}(y_n^-)$. In particular,

$$(\Gamma_+^{(n)}\cup\Gamma_-^{(n)})\cap(\{0\}\times\mathbb{R}) = \emptyset \tag{45}$$

when n is large enough, since then $\varepsilon_n < \frac{R}{4}$ but $|y_n^\pm\cdot e_1| \geq R$ (since $Q \subset S$ and $y_n^\pm \notin S$ while $q_f(y_n) \to e_1$).
Fix n and set $y := (0, f_N(0))$. For all $\eta > 0$ small enough, setting

$$D_\eta^\pm := \{(x_1,x_2) \in B_\eta(y) : \pm x_2 > \pm f_N(x_1)\}$$

we clearly have that $D_\eta^- \subset S_n$ and that D_η^+ intersects the line $Z := \{(0,x_2) : x_2 > f_N(0)\}$. On the other hand, $Z\cap\alpha_n = \emptyset$ by (45) and since by (41) Z does not intersect graph $f_N\cup$graph f_n (which contains $[y_n]\cup[y_N]$). Since Z is unbounded and connected, this implies $Z \subset U_\infty(\alpha_n)$. Hence $D_\eta^+ \subset U_\infty(\alpha_n)$ because it is connected, intersects Z but does not intersect α_n (because $B_\eta(y) \subset S$ and since D_η^+ does not intersect graph $f_n\cup$ graph f_N). But since $y \in [y_N] \subset \partial V_n$, we have $B_\eta(y)\cap V_n \neq \emptyset$. Thus $D_\eta^-\cap V_n \neq \emptyset$, so by connectedness $D_\eta^- \subset V_n$. Hence $S_n\cap V_n \neq \emptyset$, and (44) follows.

Setting $Z' := \{x + te_2 : t \leq 0\}$, we see that $x \in U_\infty(\alpha_n)$ for all n large enough, since $Z'\cap\alpha_n = \emptyset$ and Z' is unbounded and connected. So $x \notin \bar{V}_n$, whence by (44) we conclude that x is not contained in $\bar{U}$ either. But this contradicts the fact that $x \in \partial U$. (Alternatively, from (44) one could obtain a contradiction to positive area of U, since N and n were arbitrary.) Thus indeed

$$U\cap\Big([-R,R]\times(0,\delta)\Big) = \emptyset. \tag{46}$$

Hence setting $r = \min\{\delta, R\}$ we have found that U does not intersect B_r^+. So by openness of B_r^+ also $\bar{U} \cap B_r^+ = \emptyset$. This concludes the proof of Claim #3 and hence that of Claim #2. By Claim #2, the set U satisfies condition (B_f).

To prove the last statement, let U_1 and U_2 be connected components of C_f. Suppose that we had $U_1 \neq U_2$, so $U_1 \cap U_2 = \emptyset$. Let $x \in \bar{U}_1 \cap \bar{U}_2 \cap S$. Then Lemma 5 (viii) implies that $U_1 \cup [x] \cup U_2$ is open and connected. Hence by continuity of f it is contained in C_f. This contradicts maximality of U_1. □

The following proposition shows that all except finitely many connected components of C_f 'almost' belong to the set $\hat{C}_f$ introduced below. This observation is central to the proof of Theorem 3.

Proposition 8 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S; \mathbb{R}^P)$ be countably developable. Then for all $\delta > 0$ there exist $N \in \mathbb{N}$ and connected components U_k of C_f, $k = 1, ..., N$, with the following property: If U is a connected component of $C_f \setminus (U_1 \cup ... \cup U_N)$ then there exist $z_1, z_2 \in Z_U$ such that*

$$\mathcal{H}^1([z]) < \delta \text{ for all } z \in Z_U \setminus \{z_1, z_2\}. \tag{47}$$

Proof. Let $\delta > 0$ and assume for contradiction that there were an infinite sequence (U_n) of pairwise disjoint connected components of C_f violating (47) for any pair $z_1, z_2 \in Z_{U_n}$. Since by Proposition 7 and Lemma 5 the sum $\sum_{x \in Z_{U_n}} \mathcal{H}^1([x])$ converges, there is $y_n \in Z_{U_n}$ with $\mathcal{H}^1([x]) \leq \mathcal{H}^1([y_n])$ for all $x \in Z_{U_n}$. After passing to subsequences, there exists a line segment Y (in general Y cannot be written as $\overline{[x]}$, and possibly $Y \subset \partial S$) such that

$$d_{\mathcal{H}}(\overline{[y_n]}, Y) \to 0. \tag{48}$$

If we had $\mathcal{H}^1(Y) < \delta$, then by (48) we would have $\mathcal{H}^1([x]) \leq \mathcal{H}^1([y_n]) \to \mathcal{H}^1(Y) < \delta$ for all $x \in Z_{U_n}$. So (47) would hold for large n. This contradiction shows that Y must be a nondegenerate line segment with $\mathcal{H}^1(Y) \geq \delta$.
We choose coordinates such that $Y = [-2R, 2R] \times \{0\}$ for some $R > 0$. As in the proof of Proposition 7, each line $[y_n]^{\mathbb{R}^2}$ is the graph of an affine function $f_n : \mathbb{R} \to \mathbb{R}$. Arguing as in that proof, after possibly rotating the coordinates by 180 degree and passing to subsequences, by (48) we have $0 < f_{n+1} < f_n$ for all n and f_n converges to zero uniformly on $[-R, R]$. The inequality $f_n > 0$ uses the fact that $[y_n]$ does not intersect Y for n large enough. In fact, if $[y_n] \subset S$ intersected Y for infinitely many n (and $\overline{[y_n]} \not\subset Y$, but anyway $\overline{[y_n]} \subset Y$ can be true for at most finitely many $[y_n]$ because they are disjoint and their length is uniformly bounded from below), then it is easy to see that by (48) it must intersect $[y_m]$ for $m > n$ large enough, a contradiction. (This does not exclude the possibility that one endpoint $y_n^{\pm} \in \partial S$ of each $[y_n]$ agrees with an endpoint of Y.) Now by an argument similar to the one which in the proof of Proposition 7 led to $S_n \subset V_n$, one can prove that $(0, f_n(0)) \in V_n$, where $V_n := S([y_{n-1}], [y_{n+1}])$ is as furnished by Lemma 3. To apply Lemma 3 we use that, due to (48), the $[y_n]$ form a $d_{\mathcal{H}}$-Cauchy sequence, while $\mathcal{H}^1([y_n]) \geq \frac{\delta}{2}$ for large n.
But since graph $f_n|_{(-R,R)} \subset [y_n]$ (see e.g. the proof of Proposition 7), we also have $(0, f_n(0)) \in [y_n] \subset \partial U_n$. Since V_n is open, this implies that V_n intersects U_n, so by connectedness

$$U_n \subset V_n \text{ for all } n \text{ large enough,} \tag{49}$$

since U_n does not intersect $\partial V_n \subset [y_{n-1}] \cup [y_{n+1}] \cup \partial S$ because $[y_m] \subset S \setminus C_f$ (see e.g. Claim #1 in the proof of Proposition 7).
Now let $x \in Z_{U_n}$. Then by (49) the endpoints $x^{\pm}$ of $[x]$ lie in $\bar{V}_n \cap \partial S$. By Lemma 3 this set is contained in $\Gamma_+^{(n)} \cup \Gamma_-^{(n)}$, where $\Gamma_{\pm}^{(n)}$ are disjoint closed Lipschitz graphs

contained in ∂S, If $x \in Z_{U_n} \setminus A(\Gamma_-^{(n)}, \Gamma_+^{(n)})$, i.e. both x^+ and x^- lie in $\Gamma_+^{(n)}$ or both lie in $\Gamma_-^{(n)}$ then by Lemma 3,

$$\mathcal{H}^1([x]) \leq \mathcal{H}^1(\Gamma_\pm^{(n)}) \leq C d_{\mathcal{H}}([y_{n-1}], [y_{n+1}]) < \delta$$

for n large enough. But by Proposition 7 and Lemma 5 there can be at most two different $x \in Z_{U_n} \cap A(\Gamma_-^{(n)}, \Gamma_+^{(n)})$. We conclude that, for n large enough, U_n satisfies (47). This contradiction finishes the proof. □

4 The set $\hat{C}_f$ and approximation by finitely developable mappings

In this section we show that every countably developable mapping f can be approximated by 'finitely' developable mappings, i.e. countably developable mappings which consist of finitely many developable pieces and finitely many connected components of C_f. Moreover, each approximant agrees with f outside an exceptional set which is contained in an arbitrarily small neighbourhood of ∂S. This exceptional set is compatible with the level sets of f, and the approximant is constant each of its connected components. The geometric facts about C_f proven in the previous section will be essential in the proof.
Another essential observation is that every countably developable mapping is, in fact, developable on a larger set than just $S \subset C_f$. This was observed for convex domains in [14].

4.1 The set $\hat{C}_f$

Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S; \mathbb{R}^P)$ be countably developable. As observed in the introduction, it will be useful to single out those connected components U of C_f for which $\partial U \cap S$ consists of at least three connected components, i.e. $\#Z_U > 2$. Recall that $\hat{C}_f$ is the union of all connected components U of C_f for which this is the case.

The set $\hat{C}_f$ may seem to be slightly artificial. However, it turns out to be more than just a technical tool. For instance, it plays a natural role in the regularity results from [9]. The reason for its relevance is the following fact:

Proposition 9 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S; \mathbb{R}^P)$ be countably developable. Then f is S-developable on $S \setminus \hat{C}_f$. More precisely, there exists an S-ruling $\hat{q}_f : S \setminus \hat{C}_f \to \mathbb{S}^1$ such that $\hat{q}_f = q_f$ on $S \setminus C_f$ and such that f is constant on $[x]^S_{\hat{q}_f(x)}$ for all $x \in S \setminus \hat{C}_f$.*

Proof. We must show that the ruling q_f can be extended to every connected component of $C_f \setminus \hat{C}_f$. Let U be such a component. So by Proposition 7 and Lemma 5 there are x_1, $x_2 \in S \setminus C_f$ (possibly equal) such that $Z_U = \{x_1, x_2\}$, and U agrees with a connected component of $S \setminus ([x_1] \cup [x_2])$. By Claim #1 in the proof of Proposition 7 (or by condition (B_f)) we have $[x]_{q_f(x)} \cap U = \emptyset$ for all $x \in S \setminus C_f$. Therefore, if $\tilde{q}$ is an S-ruling on U then $\hat{q}$, defined by

$$\hat{q}(x) = \begin{cases} q_f(x) & \text{if } x \in S \setminus C_f \\ \tilde{q}(x) & \text{if } x \in U, \end{cases} \tag{50}$$

will be an S-ruling for f on $(S \setminus C_f) \cup U$ provided that $[x]^S_{\tilde{q}(x)} \subset U$ for all $x \in U$. To ensure this, by connectedness of $[x]^S_{\tilde{q}(x)}$ and since $S \cap \partial U = [x_1] \cup [x_2]$, we must

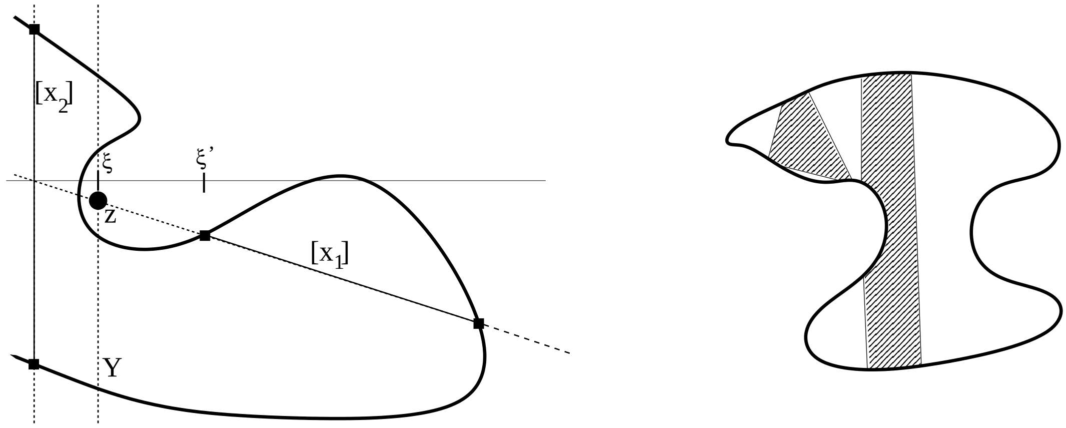

Figure 3: Left: Proof of Proposition 9. The bold curve belongs to ∂S and the black squares are the endpoints of the $[x_i]$. Right: The shaded regions are connected components $\hat{C}_f$. The mapping f is developable on parts of the left component and it is developable on the whole right component.

only make sure that $[x]^S_{\tilde{q}(x)}$ does not intersect $[x_1] \cup [x_2]$ for any $x \in U$.
Now we proceed to construct such a $\tilde{q}$. If $q_f(x_1) \parallel q_f(x_2)$ (in particular, if $x_1 = x_2$) then we define $\tilde{q}(x) = q_f(x_1)$ for all $x \in U$ and we are done.
So let us assume that $q_f(x_1)$ is not parallel to $q_f(x_2)$, so in particular $x_1 \neq x_2$. Then the intersection $[x_1]^{\mathbb{R}^2} \cap [x_2]^{\mathbb{R}^2}$ consists of a single point v (here and below we omit the subindex when it is $q_f(x_i)$). Since $[x_1] \cap [x_2] = \emptyset$ (here and below we omit the superindex when it is S), after possibly relabelling x_1 and x_2 we may assume that $v \notin [x_1]$. After rotating, translating and reflecting we may assume without loss of generality that v agrees with the origin, that $[x_2]^{\mathbb{R}^2} = \{\pi = 0\}$ and that $\pi([x_1]) \subset (0, \infty)$, where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\pi(x) = x \cdot e_1$. The condition $\pi([x_1]) \subset (0, \infty)$ can be arranged because $[x_1] \cap [x_2]^{\mathbb{R}^2} = \emptyset$ (since $v \notin [x_1]$), so $[x_1]$ does not intersect the vertical axis. Hence π does not change sign on $[x_1]$.
Set $\xi' = \inf \pi([x_1])$, so $\xi' \geq 0$. Let $\xi \in [0, \xi']$ such that

$$\left(\pi^{-1}(\xi) \cap [x_1]^{\mathbb{R}^2}\right) \setminus S \neq \emptyset. \tag{51}$$

Such ξ exists because ξ' itself satisfies (51), since the endpoints of $[x_1]$ lie in $\mathbb{R}^2 \setminus S$. We just wish to leave some more freedom; compare the remarks below.
Set $Y = \pi^{-1}(\xi)$ and denote by z the unique intersection point of Y with $[x_1]^{\mathbb{R}^2}$, which exists since $[x_1]$ is not parallel to $q_f(x_2) = \pm e_2$. By (51) we have $z \notin S$. (The situation is depicted in Figure 3 (left).) Notice that $\pi([x_2]^{\mathbb{R}^2}) = \{0\}$ and that $\pi([x_1]) \in (\xi, \infty)$ since $\xi \leq \xi'$. For $x \in \mathbb{R}^2 \setminus \{z\}$ define

$$\tilde{q}(x) = \begin{cases} q_f(x_2) & \text{if } \pi(x) \leq \xi \\ \frac{z-x}{|z-x|} & \text{otherwise.} \end{cases} \tag{52}$$

It is easy to check (see below) that $\tilde{q}$ is an $(\mathbb{R}^2 \setminus \{z\})$-ruling on $\mathbb{R}^2 \setminus \{z\}$ and that $\tilde{q}(x_i) \parallel q_f(x_i)$, $i = 1, 2$. Since $z \notin S$ we have $S \subset \mathbb{R}^2 \setminus \{z\}$, so $\tilde{q}|_S$ is an S-ruling, so in particular $[x]^S_{\tilde{q}(x)}$ does not intersect $[x_i] = [x_i]^S_{\tilde{q}(x_i)}$ for any $x \in U$ and $i = 1, 2$. Thus (50) furnishes the desired extension of q_f to U.
For the convenience of the reader let us prove that $\tilde{q}$ is a ruling on $\mathbb{R}^2 \setminus \{z\}$ with $\tilde{q}(x_i) \parallel q_f(x_i)$. Indeed, $\tilde{q}(x_2) = q_f(x_2)$ since $\pi(x_2) = 0 \leq \xi$. And $\tilde{q}(x_1)$ is parallel to

$z - x_1$, whence it is parallel to $q_f(x_1)$ because both z and x_1 lie on $[x_1]^{\mathbb{R}^2}$. Now let

$$x, y \in \mathbb{R}^2 \setminus \{z\} \text{ with } [x]_{\tilde{q}(x)}^{\mathbb{R}^2\setminus\{z\}} \cap [y]_{\tilde{q}(y)}^{\mathbb{R}^2\setminus\{z\}} \neq \emptyset. \tag{53}$$

We must show that then $[x]_{\tilde{q}(x)}^{\mathbb{R}^2\setminus\{z\}} = [y]_{\tilde{q}(y)}^{\mathbb{R}^2\setminus\{z\}}$. If $\pi(x) \leq \xi$ and $\pi(y) \leq \xi$ then $[x]_{\tilde{q}(x)}^{\mathbb{R}^2}$ and $[y]_{\tilde{q}(y)}^{\mathbb{R}^2}$ are parallel, so if they intersect then they agree. If $\pi(x) \leq \xi$ and $\pi(y) > \xi$ then $[x]_{\tilde{q}(x)}^{\mathbb{R}^2\setminus\{z\}} \cap [y]_{\tilde{q}(y)}^{\mathbb{R}^2\setminus\{z\}} = \emptyset$, so this cannot happen for x, y as in (53). Finally, if $\pi(x) > \xi$ and $\pi(y) > \xi$ then $z \in [x]_{\tilde{q}(x)}^{\mathbb{R}^2} \cap [y]_{\tilde{q}(y)}^{\mathbb{R}^2}$. But (53) implies that these two lines also intersect inside $\mathbb{R}^2 \setminus \{z\}$. So they intersect in two points and therefore they must agree. □

Remarks.

(i) If v as defined in the previous proof lies in $\mathbb{R}^2 \setminus S$ (this is always the case when S is convex) then the situation is much easier: one can simply define $\tilde{q}(x) = \frac{v-x}{|v-x|}$.

(ii) If $\xi' = 0$ then $\overline{[x_1]} \cap \overline{[x_2]} \neq \emptyset$ and $z = v$. So this is a particular case of (i). In this case, moreover, the extension $\tilde{q}$ of q_f is uniquely determined (up to signs) on U, and (in the terminology introduced later) if Γ is a $\tilde{q}$-integral curve then $s_\Gamma^* \kappa = 1$ for some $* \in \{+, -\}$.

(iii) If $\overline{[x_1]}$ intersects ∂S transversally (see the definitions in Section 5.1) and if $\xi' > 0$ then one can show that there always exists $\xi \in (0, \xi')$ satisfying (51) even with $\bar{S}$ instead of S. If this is the case, then there exists an open interval of such ξ, since $\mathbb{R}^2 \setminus \bar{S}$ is open. So the extension $\tilde{q}$ of q_f to U is not uniquely determined in this case.

(iv) The set $S \setminus \hat{C}_f$ is in general not the largest set on which f is developable: There can be components U of C_f with $\#Z_U > 2$ to which the ruling q_f can be extended (at least partially), see Figure 3 (right).

4.1.1 Approximation by finitely developable mappings.

Definition 3 *Let $S \subset \mathbb{R}^2$ be a Lipschitz domain. A mapping $f : S \to \mathbb{R}^P$ is called* finitely developable *if it is countably developable and $\hat{C}_f$ consists of finitely many connected components $U_1, ..., U_K$, and each of them satisfies $\#Z_{U_k} < \infty$ for $k = 1, ..., K$.*

Notice that if f is finitely developable then $S \setminus \hat{C}_f$ consists of finitely many connected components and f is developable on each of them by Proposition 9.

Now let f be an arbitrary countably developable mapping on S. We define

$$[x] := [x]_{\hat{q}_f(x)}^S \text{ for all } x \in S \setminus \hat{C}_f,$$

where $\hat{q}_f$ is an extension of q_f to $S \setminus \hat{C}_f$, whose existence is ensured by Proposition 9. Obviously, $[x]$ agrees with the old notation for $x \in S \setminus C_f$. We also write

$$A_{ij} := A_{ij}^{f, S\setminus\hat{C}_f}.$$

We introduce the mapping $\Omega_f : S \to 2^S$ by setting

$$\Omega_f(x) = \begin{cases} [x] & \text{if } x \in S \setminus \hat{C}_f \\ \mathcal{C}(C_f; x) & \text{if } x \in \hat{C}_f. \end{cases}$$

We define $p_f(x) = \sup \mathrm{dist}_{\partial S}(\Omega_f(x))$. For $\delta > 0$ we set

$$S_\delta^f = \{x \in S : p_f(x) > \delta\}. \tag{54}$$

We will omit the index f when it is clear from the context.

Lemma 6 *Ω_f is lower semicontinuous in the sense that $x_n \to x$ in S implies $\sup \mathrm{dist}_{\Omega_f(x_n)}(\Omega_f(x)) \to 0$. In particular, the function p_f is lower semicontinuous.*

Proof. Let $x_n, x \in S$ with $x_n \to x$. If $x \in \hat{C}_f$ then by openness $x_n \in \Omega_f(x)$ for large n, so $\Omega_f(x_n) = \Omega_f(x)$. If $x \in S \setminus \hat{C}_f$, then $\Omega_f(x) = [x]$ and $x \notin \mathrm{int}\,\Omega_f(x_n)$ for any n, so $|x - x_n| \geq \mathrm{dist}_{\Omega_f(x_n)}(x) = \mathrm{dist}_{\partial\Omega_f(x_n)}(x)$ for all n. Thus there are $x_n' \in S \cap \partial\Omega_f(x_n) \subset S \setminus C_f \subset S \setminus \hat{C}_f$ converging to x. If $x_n \notin \hat{C}_f$, then we can take $x_n' = x_n$ (since of course $\partial\Omega_f(y) = \Omega_f(y)$ for $y \in S \setminus \hat{C}_f$). From Lemma 2 (ii) and continuity of $\hat{q}_f$ we conclude that

$$\sup \mathrm{dist}_{[x_n']}([x]) \to 0. \tag{55}$$

But the whole segments $[x_n']$ are contained in $\overline{\Omega_f(x_n)}$: If $x_n \notin \hat{C}_f$ then because $[x_n'] = [x_n] = \Omega_f(x_n)$. If $x_n \in \hat{C}_f$ then we have $[x_n'] \subset \partial\Omega_f(x_n)$ because $x_n' \in \partial\Omega_f(x_n)$ and $\Omega_f(x_n)$ satisfies (B_f) by Proposition 7. Hence $\mathrm{dist}_{\Omega_f(x_n)} \leq \mathrm{dist}_{[x_n']}$, so $\sup \mathrm{dist}_{\Omega_f(x_n)}([x]) \leq \sup \mathrm{dist}_{[x_n']}([x])$, and lower semicontinuity of Ω_f follows from (55).
By lower semicontinuity of Ω_f there is $\varepsilon_n \to 0$ such that $\Omega_f(x) \subset B_{\varepsilon_n}(\Omega_f(x_n))$. Hence $\mathrm{dist}_{\partial S}(\Omega_f(x)) \subset B_{\varepsilon_n}\big(\mathrm{dist}_{\partial S}(\Omega_f(x_n))\big)$. Taking suprema and passing to the limes inferior yields $p_f(x) \leq \liminf p_f(x_n)$. $\square$

Lemma 7 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain, let $f \in C^0(S; \mathbb{R}^P)$ be countably developable and let $\delta > 0$ be small. Then S_δ^f is open and connected, and*

$$\sup \mathrm{dist}_{\partial S}(S \setminus S_\delta^f) \leq \delta. \tag{56}$$

In addition, $\partial S_\delta^f \cap \hat{C}_f = \emptyset$ and $S \setminus (S_\delta^f \cup \hat{C}_f) \subset \bigcup_{i=0}^{N_S} A_{ii}$. Moreover, S_δ^f satisfies condition (B_f).

Remark.

(i) Since S_δ^f satisfies condition (B_f), we can apply Lemma 5 with $U = S_\delta^f$ to obtain a set $Z_{S_\delta^f}$ with the properties stated Lemma 5.

(ii) The implication
$$x \in S_\delta^f \setminus \hat{C}_f \implies \mathcal{H}^1([x]) > \delta \tag{57}$$
is clear from the definition. The converse implication is false in general.

(iii) In view of (57), Proposition 8 implies that
$$\#(Z_U \cap S_\delta^f) \leq 2$$
for all except finitely many connected components U of C_f.

Proof. We omit the index f. Openness of S_δ follows from lower semicontinuity of p_f, and $p_f \geq \mathrm{dist}_{\partial S}$ implies (56). Another immediate consequence of the definition is that

$$S_\delta = \{x \in S : \Omega_f(x) \subset S_\delta\} \text{ and } S \setminus S_\delta = \{x \in S : \Omega_f(x) \cap S_\delta = \emptyset\}. \tag{58}$$

To prove connectedness of S_δ, let $x, y \in S_\delta$. Hence there are $x' \in \Omega_f(x)$ and $y' \in \Omega_f(y)$ such that $x', y' \in S \setminus \overline{B_\delta(\partial S)}$. But for δ small enough the latter set is connected by Lemma 18 in the appendix. So there is a continuous curve inside $S \setminus \bar{B}_\delta(\partial S)$, so by (56) inside S_δ, which connects x' and y'. Since x and x' lie in the connected set $\Omega(x) \subset S_\delta$ and y and y' in the connected set $\Omega(y) \subset S_\delta$, we conclude that S_δ is connected.
To prove $\partial S_\delta \cap S \subset S \setminus \hat{C}_f$, suppose there were $x \in \hat{C}_f \cap \partial S_\delta$. Then for all $\varepsilon > 0$ we would have $B_\varepsilon(x) \cap S_\delta \neq \emptyset$, but for ε small enough we have $B_\varepsilon(x) \subset \Omega_f(x)$, so $\Omega_f(x) \cap S_\delta \neq \emptyset$. Hence $x \in S_\delta$ by (58), contradicting openness.
Next, let $x \in \partial S_\delta \cap S \subset S \setminus \hat{C}_f$. We claim that then $[x] \subset \partial S_\delta$. Since S_δ is open, $x \notin S_\delta$, so by (58) we have $[x] \cap S_\delta = \emptyset$. Hence it suffices to show $[x] \subset \bar{S}_\delta$. But since $x \in \partial S_\delta$ there exist $x_n \in S_\delta$ converging to x. Hence by lower semicontinuity of Ω_f (see Lemma 6) we have $[x] = \Omega_f(x) \subset B_{\varepsilon_n}(\Omega_f(x_n))$ for some $\varepsilon_n \to 0$. This implies the claim because $\Omega_f(x_n) \subset S_\delta$ by (58).
Finally notice that $S \setminus (S_\delta \cup \hat{C}_f) \subset \bigcup_{i=0}^{N_S} A_{ii}$ is an immediate consequence of (56) and of Lemma 2 (vi). □

Now we are ready to prove the main result of this section; it is a precise version of Theorem 1 stated in the Introduction.

Theorem 3 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S; \mathbb{R}^P)$ be countably developable. Then there is $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the following is true:*
The set $S \setminus \overline{S}^f_\delta$ consists of countably many connected components $W_1, W_2, \ldots$ which satisfy

$$\bar{W}_i \cap \bar{W}_j \cap S = \emptyset \text{ whenever } i \neq j,$$

and for all i there is $x_i \in S \setminus \hat{C}_f$ such that

$$\bar{S}^f_\delta \cap \bar{W}_i \cap S = [x_i].$$

The mapping $f_\delta : S \to \mathbb{R}^P$ defined by

$$f_\delta(x) = \begin{cases} f(x) & \text{if } x \in S \cap \bar{S}^f_\delta \\ f(x_i) & \text{if } x \in S \cap \bar{W}_i \text{ for some } i. \end{cases} \tag{59}$$

is well-defined, uniformly bounded and continuous on S, and it is finitely developable.

Proof. We omit the index f and we choose $\delta > 0$ small enough to satisfy the hypothesis of Lemma 7. Set $f_\delta := f$ on $S \cap \bar{S}_\delta$. As in Lemma 5, for $x \in Z_{S_\delta}$ denote by S^2_x the connected component of $S \setminus [x]$ that does not intersect S_δ. For each $x \in Z_{S_\delta}$ we set f_δ constantly equal to $f(x)$ on S^2_x. By (27) (which holds for S_δ instead of U due to Lemma 7) this determines f_δ on all of S. Moreover, f_δ is well defined because $S^2_x \cap S^2_y = \emptyset$ if $x, y \in Z_{S_\delta}$ with $x \neq y$, see Lemma 5 (v). Clearly, f_δ is continuous, since so is f. By construction we have $\bigcup_{x \in Z_{S_\delta}} S^2_x \subset C_{f_\delta}$. Thus by (27) we conclude

$$S \setminus \bar{S}_\delta \subset C_{f_\delta}. \tag{60}$$

In particular, f_δ is countably developable, since so is f, and $C_f \subset C_{f_\delta}$. It remains to show that f_δ is finitely developable.

Claim #1. Let U' be a connected component of C_{f_δ}. If $\#Z_{U'} > 1$ (with the notation of Proposition 7 applied to f_δ) then $\inf\{\mathcal{H}^1([x]) : x \in Z_{U'}\} \geq \delta$.

To prove this, assume that there is $x \in Z_{U'} \subset S \setminus C_{f_\delta} \subset S \setminus C_f$ with $\mathcal{H}^1([x]) < \delta$. Then $[x] \in S \setminus S_\delta$ by the definition of S_δ. On the other hand, by (60) and since

$x \in S \cap \partial U' \subset S \setminus C_{f_\delta}$, we have $x \in S \cap \bar{S}_\delta$. Thus $x \in S \cap \partial S_\delta$ and so $[x] \subset \partial S_\delta$ because S_δ satisfies condition (B_f) by Lemma 7.
As seen above, $S_x^2 \subset C_{f_\delta}$ (to use this notation we assume without loss of generality that x agrees with the unique element in $[x] \cap Z_{S_\delta}$). So S_x^2 is contained in a connected component U_1 of C_{f_δ}. In particular $x \in \bar{U}_1$ because $[x] \subset \bar{S}_x^2$ (e.g. by Lemma 16 (i)). Thus by the last part of Proposition 7, applied with $U_2 = U'$, we have $U' = U_1$ since $x \in S \cap \bar{U}' \cap \bar{U}_1$. Since $S_x^2 \subset U_1$, we have $S_x^2 \cap U_1 \neq \emptyset$. But $\partial S_x^2 \subset \partial S \cup [x]$ by Lemma 16 (i), and U' does not intersect this set. Thus by connectedness $U_1 \subset S_x^2$. We conclude that $U' = U_1 = S_x^2$, i.e. U' agrees with a connected component of $S \setminus [x]$. So $S \cap \partial U' = [x]$, whence $\#Z_{U'} = 1$. This proves the claim.
Now let U' be a connected component of $\hat{C}_{f_\delta}$. Then by Claim #1 we have

$$\inf_{x \in Z_{U'}} \mathcal{H}^1([x]) \geq \delta. \tag{61}$$

Since $\sum_{x \in Z_{U'}} \mathcal{H}^1([x]) < \infty$ by Proposition 7 and Lemma 5, this implies $\#Z_{U'} < \infty$. Moreover, by Proposition 8 there exist only finitely many components U' of $\hat{C}_{f_\delta}$ satisfying (61). Thus f_δ is finitely developable.
In order to recover the notation used in the statement of the theorem, denote the elements of Z_{S_δ} by $x_1, x_2, ...$ and define $W_i = S_{x_i}^2$.
Finally, we show that f_δ as constructed above belong to L^∞. In fact, formula (59) implies that

$$f_\delta(S) \subset f(S_\delta). \tag{62}$$

Set $D_\delta = \{x \in S : \mathrm{dist}_{\partial S}(x) > \delta\}$. Then S_δ consists, by definition, of all connected components U of $\hat{C}_f$ with $U \cap D_\delta \neq \emptyset$ and all $x \in S \setminus \hat{C}_f$ such that $[x] \cap D_\delta \neq \emptyset$. Since f is constant on such U and on such $[x]$, it follows that

$$f(S_\delta) = f(D_\delta),$$

which is a bounded set because f is continuous on $\bar{D}_\delta$. Hence (62) shows that indeed $f_\delta \in L^\infty(S; \mathbb{R}^P)$. □

5 The developable part

In the previous sections we studied the set C_f of local constancy of a countably developable mapping f. In this section we will study its complement. By definition, f is developable on this set; thus, the level sets of f are straight line segments which do not intersect inside S. A very natural way to label these segments is to introduce an arclength parametrized curve that runs perpendicular to them:

Remark 3 *Let $x_0 \in \mathbb{R}^2$ and $R > 0$, and let $f : B_R(x_0) \to \mathbb{R}^P$ be $B_R(x_0)$-developable. Then there exist unique $t_0 \leq -R$ and $t_1 \geq R$ and a unique curve $\Gamma \in W^{2,\infty}_{loc}((t_0, t_1); B_R(x_0))$ solving the ordinary differential equation*

$$\Gamma'(t) = -q_f^\perp(\Gamma(t)) \tag{63}$$

with $\Gamma(0) = x_0$ and $\Gamma(t_k) \in \partial B_R(x_0)$, $k = 0, 1$.

Proof. By Remark 1, after appropriately choosing antipodal points we have $q_f \in W^{1,\infty}_{loc}(B_R(x_0); \mathbb{S}^1)$. Hence the claim follows from standard existence theory for ordinary differential equations. The lower bounds on $|t_0|$, $|t_1|$ follow from the fact that $|\Gamma'| = 1$ on (t_0, t_1). □

We remark that, when f is the gradient ∇u of an isometric immersion $u : S \to \mathbb{R}^3$, then a curve as in Remark 3 is a line of curvature of u. In that geometric setting it is natural to consider the line of curvature chart defined by Γ. The same can be done in the abstract context considered here: We will associate with a given curve $\Gamma : [0, T] \to S$ a set $[\Gamma(0, T)]$ which can be parametrized by (generalized) line of curvature parameters determined by Γ. The corresponding change of variables is given by $\Phi_\Gamma(s, t) = \Gamma(t) + sq_f(\Gamma(t))$, where the admissible values of s depend on $t \in [0, T]$. In this section we study the relation between invertibility properties of Φ_Γ and certain 'admissibility' properties of Γ. We will also study the regularity of the patch $[\Gamma(0, T)]$ which is parametrized by Φ_Γ.

Now follows an overview over this chapter. In Section 5.1 we introduce the directed distance function ν^S, which provides a link between analytic and topological properties of f. Moreover, we make precise the notions mentioned above, and we isolate certain necessary properties which a curve Γ must satisfy in order to serve as a (generalized) line of curvature. In Section 5.2 we study the analytic properties of the change of variables Φ_Γ, and we establish some links between its invertibility properties and some properties of Γ. In Section 5.4 we prove the regularity properties of the directed distance function ν^S mentioned in the introduction. In Section 5.3 we address this question: Under which circumstances can f be extended, as a countably developable mapping, to a larger domain? The results will not be used in this paper, but they are used in [10] in order to answer the analogous question in the context of isometric immersions. In Section 5.6 we obtain an important global regularity result about the directed distance function along suitable curves, proving Corollary 1. Finally, in Section 5.7 we introduce the notion of f-integral curves, which are essentially those curves obtained in Remark 16, and we prove some regularity properties about the relative boundary $S \cap \partial[\Gamma(0, T)]$ if Γ is such a curve. These are important in the proof of the decomposition theorem (Theorem 4) in the next chapter.

5.1 Admissibility, local admissibility and the directed distance

From now on, $T > 0$ and $\Gamma \in W^{2,\infty}((0, T); \mathbb{R}^2)$ denotes a curve with $|\Gamma'(t)| = 1$ for all $t \in [0, T]$ and curvature $\kappa \in L^\infty(0, T)$. We define $N := (\Gamma')^\perp$; thus

$$\kappa = \Gamma'' \cdot N.$$

We define the Frénet frame $R := (\Gamma', N)^T$. The Frénet equations read

$$R' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} R. \tag{64}$$

We now introduce a mapping which will eventually define a natural chart associated with Γ. In the case of isometric immersions, it will turn out to be a local line of curvature parametrization. Define the mapping

$$\Phi_\Gamma : \mathbb{R} \times [0, T] \to \mathbb{R}^2$$

by

$$\Phi_\Gamma(s, t) := \Gamma(t) + sN(t).$$

Much of the following discussion will focus on the relation between the injectivity of Φ_Γ (on an appropriate domain) and properties of the curve Γ. For a given curve Γ, the actual domain on which Φ_Γ will be studied will be of the form

$$M_{s^\pm} := \bigcup_{t \in (0,T)} (s^-(t), s^+(t)) \times \{t\} \tag{65}$$

for given $s^{\pm} : [0,T] \to \mathbb{R}$, with $s^-(t) < 0 < s^+(t)$ for all $t \in [0,T]$.

We will be mainly concerned with curves Γ taking values inside a given domain S; in this case, there is a natural choice of $s^{\pm}$ by which they will be determined by Γ and S. However, it will sometimes be useful to view the curve Γ as being independent of any given domain S. This is certainly more natural from the viewpoint of the mapping Φ_Γ. We therefore start by introducing, somewhat abstractly, the following notions:

Definition 4 *Let $\pm s^{\pm} : [0,T] \to \mathbb{R}$ be Borel functions which are bounded from above and from below by positive constants. We define (abusing notation)*

$$[\Gamma(t)] := \{\Gamma(t) + sN(t) : s \in (s^-(t), s^+(t))\}. \tag{66}$$

For $J \subset [0,T]$ we introduce the notation

$$[\Gamma(J)] = \bigcup_{t \in J} [\Gamma(t)].$$

We also set

$$\beta_\Gamma^{\pm} := \Gamma + s^{\pm} N.$$

(i) Γ is called locally $s^{\pm}$-admissible if $1 - s^{\pm}(t)\kappa(t) \geq 0$ for almost every $t \in (0,T)$.

(ii) Γ is called $s^{\pm}$-admissible if $[\Gamma(t_1)] \cap [\Gamma(t_2)] = \emptyset$ for all $t_1, t_2 \in [0,T]$ with $t_1 \neq t_2$.

(iii) Γ is called uniformly locally $s^{\pm}$-admissible if there is $c > 0$ such that $1 - s^{\pm}(t)\kappa(t) \geq c$ for almost every $t \in (0,T)$.

(iv) Γ is called uniformly $s^{\pm}$-admissible if it is $s^{\pm}$-admissible and uniformly locally $s^{\pm}$-admissible.

Let $S \subset \mathbb{R}^2$ be a domain. We recall that the *directed distance function*

$$\nu^S : S \times (\mathbb{R}^2 \setminus \{0\} \to (0,\infty)$$

is defined by

$$\nu^S(x,\mu) = \inf\{\theta > 0 : x + \theta\mu \notin S\} \text{ for all } (x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\}). \tag{67}$$

In what follows, we will often omit the index S. Now we introduce the announced natural choice for $s^{\pm}$ for curves taking values in S: If $\Gamma \in W^{2,\infty}([0,T];S)$ then we define

$$s_\Gamma^{\pm} := \pm\nu(\Gamma, \pm N)$$

and

$$\beta_\Gamma^{\pm} = \Gamma + s_\Gamma^{\pm} N,$$

and we define $[\Gamma(t)]$ as in (66) with $s_\Gamma^{\pm}$ instead of $s^{\pm}$. Notice that $[\Gamma(t)] = [\Gamma(t)]_{N(t)}^S$ in our earlier notation. The curve Γ is said to be S-admissible (locally S-admissible, uniformly locally S-admissible, uniformly S-admissible) if it is $s_\Gamma^{\pm}$-admissible (locally $s_\Gamma^{\pm}$-admissible, uniformly locally $s_\Gamma^{\pm}$-admissible, uniformly $s_\Gamma^{\pm}$-admissible). Again, we will often omit the prefix S.

5.2 Properties of the change of variables Φ_Γ: Relation between (local) $s^{\pm}$-admissibility of Γ and (local) injectivity of Φ_Γ

The main result of this section is Proposition 12. It relates the above admissibility notions of Γ to (local) invertibility of Φ_Γ.
Given $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$, we define the auxiliary function

$$\sigma_\Gamma : (0,T) \times (0,T) \to \mathbb{R} \cup \{\infty\}$$

by setting

$$\sigma_\Gamma(t_0,t_1) = \begin{cases} \frac{1}{\kappa(t_0)} & \text{if } t_0 = t_1 \text{ and } \kappa(t_0) \neq 0 \\ \tilde{\sigma}_{t_0,t_1} & \text{if } N(t_0) \text{ is not parallel to } N(t_1). \\ \infty & \text{otherwise.} \end{cases} \tag{68}$$

If $N(t_0)$ is not parallel to $N(t_1)$ then $\tilde{\sigma}_{t_0,t_1}$ denotes the unique solution of the inclusion $\Gamma(t_0) + \tilde{\sigma}_{t_0,t_1} N(t_0) \in [\Gamma(t_1)]^{\mathbb{R}^2}_{N(t_1)}$. In what follows we will often omit the subindex Γ in the symbols just introduced.

It is easy to check that a curve Γ is $s^{\pm}$-admissible if and only if for all $t_0, t_1 \in [0,T]$ with $t_0 \neq t_1$ we have

$$\sigma_\Gamma(t_0,t_1) \notin (s^-(t_0), s^+(t_0)) \text{ or } \sigma_\Gamma(t_1,t_0) \notin (s^-(t_1), s^+(t_1)). \tag{69}$$

Equivalently,

$$\frac{1}{\sigma_\Gamma(t_0,t_1)} \in \left[\frac{1}{s^-(t_0)}, \frac{1}{s^+(t_0)}\right] \text{ or } \frac{1}{\sigma_\Gamma(t_1,t_0)} \in \left[\frac{1}{s^-(t_1)}, \frac{1}{s^+(t_1)}\right]. \tag{70}$$

Lemma 8 *Let $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$ and let $t,t' \in (0,T)$ be such that $N(t)$ is not parallel to $N(t')$. Then $N(t') \cdot \Gamma'(t) \neq 0$ and*

$$\sigma_\Gamma(t',t) = \frac{(\Gamma(t) - \Gamma(t')) \cdot \Gamma'(t)}{N(t') \cdot \Gamma'(t)}. \tag{71}$$

Moreover, there is a constant C_1 depending only on $\|\kappa\|_\infty$ such that

$$\Big|\frac{1}{\sigma_\Gamma(t',t)} - \fint_{t'}^{t} \kappa \, ds\Big| \leq C_1|t-t'|. \tag{72}$$

In particular, if t' is a Lebesgue point of κ, then $\frac{1}{\sigma_\Gamma(t',\cdot)}$ and $\frac{1}{\sigma_\Gamma(\cdot,t')}$ are continuous with $\frac{1}{\sigma_\Gamma(t',t')} = \kappa(t')$.

Proof. Clearly, $N(t)$ and $N(t')$ are not parallel if and only if $N(t') \cdot \Gamma'(t) \neq 0$. By definition we have $\Gamma(t') + \sigma_\Gamma(t',t)N(t') = \Gamma(t) + \sigma_\Gamma(t,t')N(t)$, from which (71) follows upon scalar multiplication with $\Gamma'(t)$. From (71) and using $\Gamma'' = \kappa N$ one can readily check (72):

$$\begin{aligned} \Big|\frac{1}{\sigma_\Gamma(t',t)} - \fint_{t'}^{t} \kappa\Big| &= \Big|\frac{N(t') \cdot \fint_{t'}^{t} \kappa N}{\Gamma'(t) \cdot \fint_{t'}^{t} \Gamma'} - \fint_{t'}^{t} \kappa\Big| \\ &\leq \|\kappa\|_\infty \Big(\Big|\frac{1}{\Gamma'(t) \cdot \fint_{t'}^{t} \Gamma'} - 1\Big| + \fint_{t'}^{t} |1 - N(t') \cdot N|\Big). \end{aligned}$$

The last line is clearly bounded by $C|t-t'|$ for some C depending only on $\|\kappa\|_\infty$. $\square$

Lemma 9 *Let $\pm s^{\pm} : [0,T] \to \mathbb{R}$ be bounded, lower semicontinuous and bounded from below by a positive constant, and let $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$. If there is $\eta \geq 0$ such that*

$$\kappa(t) \in \left[\frac{1}{s^-(t)} + \eta, \frac{1}{s^+(t)} - \eta\right] \text{ for almost every } t \in (0,T), \tag{73}$$

then the following are true:

(i) For all $\delta > 0$ and for all $t' \in [0,T]$ there is $\delta_1 > 0$ such that

$$\frac{1}{\sigma_\Gamma(t',t)} \in \left[\frac{1}{s^-(t')} + \eta - \delta, \frac{1}{s^+(t')} - \eta + \delta\right] \tag{74}$$

for all $t \in [0,T]$ with $|t - t'| \in (0, \delta_1]$. If $s^{\pm} \in C^0([0,T])$ then δ_1 can be chosen independently of t'.

(ii) If $s^{\pm} \in C^0([0,T])$ and Γ is uniformly locally $s^{\pm}$-admissible (i.e. $\eta > 0$ in (73)) then Φ_Γ is locally injective on $\bar{M}_{s^{\pm}}$. More precisely, there are $\delta_1, \varepsilon > 0$ such that, whenever $t_1, t_2 \in [0,T]$ with $|t_1 - t_2| \leq \delta_1$ and $s_i \in (s^-(t_i) - \varepsilon, s^+(t_i) + \varepsilon)$ ($i = 1,2$) then

$$\Phi_\Gamma(s_1,t_1) = \Phi_\Gamma(s_2,t_2) \implies (s_1,t_1) = (s_2,t_2).$$

In particular, if $(s_i,t_i) \in M_{s^{\pm}}$ and $\Phi_\Gamma(s_1,t_1) = \Phi_\Gamma(s_2,t_2)$, then either $(s_1,t_1) = (s_2,t_2)$ or $|t_1 - t_2| > \delta_1$.

Proof. We omit the index Γ. Let $t', t \in [0,T]$. We have

$$\begin{aligned} \frac{1}{\sigma(t',t)} - \frac{1}{s^+(t')} \leq \Big| &\frac{1}{\sigma(t',t)} - \fint_{t'}^{t} \kappa(r)\, dr \Big| \\ &+ \fint_{t'}^{t} \kappa(r) - \frac{1}{s^+(r)}\, dr + \fint_{t'}^{t} \frac{1}{s^+(r)} - \frac{1}{s^+(t')}\, dr. \end{aligned}$$

Since $\kappa \leq \frac{1}{s^+} - \eta$ the second term does not exceed $-\eta$, and by Lemma 8 the first term does not exceed $C_1|t - t'|$ for some C_1 depending only on $\|\kappa\|_{L^\infty(0,T)}$. Hence

$$\limsup_{t \to t'} \left(\frac{1}{\sigma(t',t)} - \frac{1}{s^+(t')}\right) \leq \limsup_{t \to t'} \left(\fint_{t'}^{t} \frac{1}{s^+(r)} - \frac{1}{s^+(t')}\, dr\right) - \eta \leq -\eta. \tag{75}$$

The last inequality follows from lower semicontinuity of s^+. By (75), for all $\delta > 0$ and for all t' there is $\delta_1 > 0$ such that

$$\frac{1}{\sigma(t',t)} \leq \frac{1}{s^+(t')} + \delta - \eta \text{ for all } t \in [t' - \delta_1, t' + \delta_1].$$

The analogous statement involving s^- is proven similarly. If $1/s^{\pm}$ are (uniformly) continuous then the size of $\fint_{t'}^{t} \frac{1}{s^+(r)} - \frac{1}{s^+(t')}\, dr$ depends only on $|t - t'|$, so δ_1 can be chosen independently of t'.
To prove the final statement, notice that by the assumptions on $s^{\pm}$ and since η is strictly positive, there are $\varepsilon, \delta > 0$ such that $\frac{1}{s^-} + \eta - \delta > \frac{1}{s^- - \varepsilon}$ and $\frac{1}{s^+} + \delta - \eta < \frac{1}{s^+ + \varepsilon}$. Thus (74) implies that $\Gamma(t') + sN(t')$ is not contained in $[\Gamma(t)]^{\mathbb{R}^2}$ for any $s \in [s^-(t') - \varepsilon, s^+(t') + \varepsilon]$ and any t with $|t - t'| \leq \delta_1$ (and by the above δ_1 does not depend on t'). Hence the claim follows from the definition of Φ. $\square$

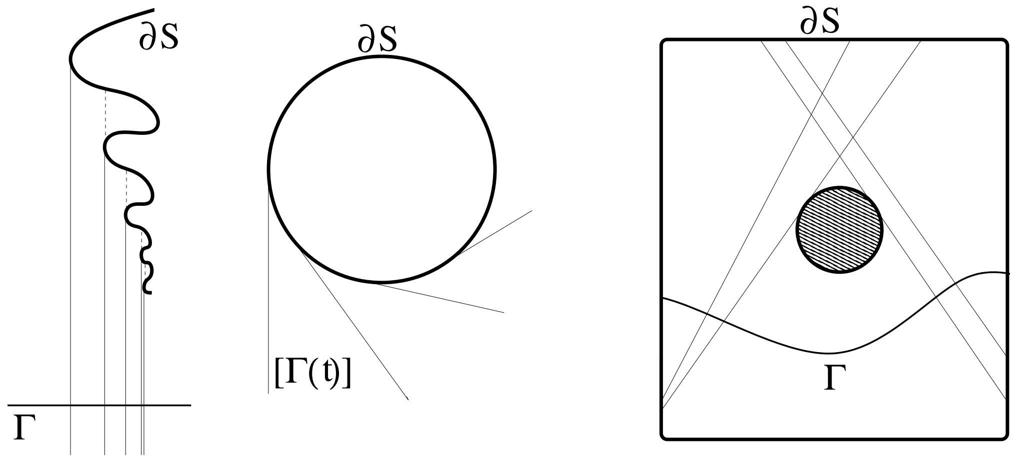

Figure 4: Left: The functions s^*_Γ, β^*_Γ are in general not continuous. Middle: The set D^*_Γ can strictly contain the jump set $A_{s^\pm_\Gamma}$ of s^*_Γ (the circle is part of ∂S). Right: The shaded disk belongs to $\mathbb{R}^2 \setminus S$. The curve Γ is locally admissible but not admissible. Hence continuity of $s^\pm$ cannot be dropped from the hypothesis of Proposition 12 (ii).

Proposition 10 *Let $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$ and let $\pm s^\pm : [0,T] \to \mathbb{R}$ be bounded, lower semicontinuous and bounded from below by a positive constant. Then the set $M_{s^\pm}$ defined in (65) is open, and $\Phi_\Gamma(M_{s^\pm}) = [\Gamma(0,T)]$. The mapping Φ_Γ is locally Lipschitz on $\mathbb{R} \times (0,T)$ and*

$$\det \nabla\Phi_\Gamma(s,t) = -(1 - s\kappa(t)) \text{ for almost every } (s,t) \in \mathbb{R} \times (0,T). \tag{76}$$

In particular, $\overline{\Phi_\Gamma(M_{s^\pm})} = \Phi_\Gamma(\bar{M}_{s^\pm})$. In addition, defining

$$\overline{s}^+(t) := \limsup_{[0,T]\ni t'\to t} s^+(t') \text{ and } \underline{s}^-(t) := \liminf_{[0,T]\ni t'\to t} s^-(t'), \tag{77}$$

and

$$A_{s^\pm} := \{t \in [0,T] : s^\pm \text{ is not continuous at } t\}, \tag{78}$$

we have

$$\bar{M}_{s^\pm} = \bigcup_{t\in[0,T]} [\underline{s}^-(t), \overline{s}^+(t)] \times \{t\} \tag{79}$$

$$\partial M_{s^\pm} = \Big(\bigcup_{t\in[0,T]} \{s^-(t), s^+(t)\} \times \{t\}\Big) \cup \Big(\bigcup_{t\in\{0,T\}} (s^-(t), s^+(t)) \times \{t\}\Big) \cup$$
$$\cup \Big(\bigcup_{t\in A_{s^+}} (s^+(t), \overline{s}^+(t)] \times \{t\}\Big) \cup \Big(\bigcup_{t\in A_{s^-}} [\underline{s}^-(t), s^-(t)) \times \{t\}\Big). \tag{80}$$

Moreover, the following hold:

(i) If Γ is locally $s^\pm$-admissible then $\det\nabla\Phi_\Gamma < 0$ *almost everywhere on $M_{s^\pm}$ and $\Phi_\Gamma|_{M_{s^\pm}}$ is an open mapping. In particular, $\Phi_\Gamma(M_{s^\pm})$ is open and*

$$\partial\Phi_\Gamma(M_{s^\pm}) \subset \Phi_\Gamma(\partial M_{s^\pm}).$$

(ii) If Γ is $s^\pm$-admissible then Γ is locally $s^\pm$-admissible. Moreover, $[\Gamma(0,T)]$ is open, $\Phi_\Gamma : M_{s^\pm} \to [\Gamma(0,T)]$ is a homeomorphism and Φ_Γ^{-1} is locally Lipschitz on $[\Gamma(0,T)]$. In fact, Φ_Γ is injective on the set

$$M_{s^\pm} \cup \Big((s^-(0), s^+(0)) \times \{0\} \cup (s^-(T), s^+(T)) \times \{T\}\Big).$$

(iii) If Γ is uniformly $s^{\pm}$-admissible then, moreover, $\Phi_\Gamma^{-1} \in W^{1,\infty}([\Gamma(0,T)];\mathbb{R}^2)$.

(iv) If Γ is $s^{\pm}$-admissible and $\Gamma'(t)\cdot\Gamma'(t') > 0$ for all $t,t' \in [0,T]$, then Φ_Γ is injective on

$$\hat{M}_{s^\pm} := \bigcup_{t\in[0,T]} (\underline{s}^-(t), \overline{s}^+(t)) \times \{t\}. \tag{81}$$

Moreover, $(\Phi_\Gamma)^{-1}$ is continuous on $\Phi_\Gamma(\hat{M}_{s^\pm})$.

Remarks.

(i) Some extra hypothesis like the one about Γ' is needed for the conclusion of (iv).

(ii) The inclusion $\partial\Phi(M_{s^\pm}) \subset \Phi(\partial M_{s^\pm})$ is strict in general.

Proof. We omit the indices Γ and $s^{\pm}$. Notice that M is open because $*s^*$ are lower semicontinuous. The expression (79) for $\bar{M}$ is clear from the definition. By openness we have $\partial M = \bar{M} \setminus M$, and since $s^+(t) > s^-(t) \geq \limsup s^-(t')$ and $s^-(t) < s^+(t) \leq \liminf s^+(t')$, equation (80) is immediate as well.
The equality $\Phi(M) = [\Gamma(0,T)]$ is just the definition. The mapping Φ is clearly locally Lipschitz on $\mathbb{R}\times(0,T)$, and Φ is continuous up to the boundary of M. From this one immediately deduces that $\overline{\Phi(M)} = \Phi(\bar{M})$. The expression (76) is immediate from (64).
To prove (i) let us assume that Γ is locally $s^{\pm}$-admissible. Since $\kappa \in [\frac{1}{s^-}, \frac{1}{s^+}]$, equation (76) implies that $\det\nabla\Phi < 0$ almost everywhere on M. More precisely, if B is a ball whose closure is contained in M then by local admissibility there is $\delta > 0$ such that $\det\nabla\Phi(s,t) = -(1-s\kappa(t)) < -\delta$ for almost every $(s,t)\in B$. (This follows from boundedness of $|s^{\pm}|$ and since $s \in (s^-(t)+\varepsilon, s^+(t)-\varepsilon)$ for all $(s,t)\in B$, where $\varepsilon := \frac{1}{2}\operatorname{dist}(B, \mathbb{R}^2\setminus M)$.) Therefore, Φ is of bounded distortion on B, since it is Lipschitz on B. Thus by [21] Theorem 6.4 we conclude that $\Phi|_B$ is an open mapping (it is clearly not constant). Since this is true for all such balls B, we conclude that $\Phi|_M$ is an open mapping. In particular, $\Phi(M)$ is open. Thus by the above $\partial\Phi(M) = \overline{\Phi(M)}\setminus\Phi(M) = \Phi(\bar{M})\setminus\Phi(M) \subset \Phi(\bar{M}\setminus M) = \Phi(\partial M)$. This concludes the proof of (i).

To prove (ii), assume that Γ is $s^{\pm}$-admissible. We claim that then Γ is also locally $s^{\pm}$-admissible. In fact, let $t_1 \in (0,T)$ be a Lebesgue point of κ. By Lemma 8 the functions $\frac{1}{\sigma(t_1,\cdot)}$ and $\frac{1}{\sigma(\cdot,t_1)}$ are continuous at t_1 and both equal $\kappa(t_1)$ at this point. Letting $t_0 \to t_1$ in (70) and using the upper semicontinuity of the functions $\pm\frac{1}{s^\pm}$, the claim follows.

By the definition of $s^{\pm}$-admissibility, of M and of $[\Gamma(0,T)]$, we have that Φ is injective on $M \cup \Big((s^-(0), s^+(0))\times\{0\} \cup (s^-(T), s^+(T))\times\{t\}\Big)$. So $\Phi(M)$ is open by the invariance of domain theorem, see e.g. Theorem 3.30 in [6]. (Of course this is also a consequence of (i).) And Φ^{-1} is well defined on $[\Gamma(0,T)]$. Now Φ is Lipschitz with $\det\nabla\Phi(s,t) < 0$ almost everywhere on M by local $s^{\pm}$-admissibility. So Theorem 6.1 in [6] (of course it does not matter whether the Jacobian is positive or negative)implies that $\Phi^{-1} \in W^{1,1}_{loc}([\Gamma(0,T)];\mathbb{R}^2)$ and that

$$\nabla(\Phi^{-1})(x) = \Big(\nabla\Phi(\Phi^{-1}(x))\Big)^{-1} \text{ for almost every } x \in [\Gamma(0,T)]. \tag{82}$$

Since Φ is continuous up to the boundary of M, its inverse Φ^{-1} is easily seen to be continuous on $[\Gamma(0,T)]$. So Φ^{-1} maps a small enough ball around $x \in [\Gamma(0,T)]$ into a small neighbourhood U of $\Phi^{-1}(x)$ with $\bar{U} \subset M$. And $(\nabla\Phi)^{-1}$ is uniformly bounded on U by (76); compare the proof of part (i). Hence (82) implies that $\Phi^{-1} \in W^{1,\infty}_{loc}([\Gamma(0,T)];\mathbb{R}^2)$. If Γ is uniformly $s^{\pm}$-admissible then $\Phi^{-1} \in$

$W^{1,\infty}([\Gamma(0,T)];\mathbb{R}^2)$ by (76, 82), but it need not be Lipschitz if $[\Gamma(0,T)]$ is not regular enough.

To prove (iv), using the hypothesis on Γ' one easily proves that

$$(\Gamma(t)-\Gamma(t'))\cdot\Gamma'(t)\neq 0 \text{ for all } t,t'\in[0,T] \text{ with } t\neq t'. \tag{83}$$

Now let $(s_m,t_m)\in\hat{M}$, $m=1,2$, be such that $\Phi(s_1,t_1)=\Phi(s_2,t_2)$. It suffices to prove $t_1=t_2$ because then clearly also $s_1=s_2$. Let us assume for contradiction that $t_1\neq t_2$. Since $[\Gamma(t_1)]^{\mathbb{R}^2}_{N(t_1)}$ and $[\Gamma(t_2)]^{\mathbb{R}^2}_{N(t_2)}$ intersect (in the point $\Phi(s_1,t_1)$), if $N(t_1)$ were parallel to $N(t_2)$ then $\Gamma(t_1)-\Gamma(t_2)$ would have to be parallel to $N(t_1)$. This would contradict (83). Thus $N(t_1)$ is not parallel to $N(t_2)$ and we can apply Lemma 8. Since $[\Gamma(t_1)]^{\mathbb{R}^2}_{N(t_1)}$ intersects $[\Gamma(t_2)]^{\mathbb{R}^2}_{N(t_2)}$ in $\Gamma(t_1)+s_1N(t_1)=\Gamma(t_2)+s_2N(t_2)$, we conclude that $s_1=\sigma(t_1,t_2)$ and $s_2=\sigma(t_2,t_1)$. From (71) and (83) we deduce that $s_1\neq 0\neq s_2$.
By definition of $\overline{s}^+$ and $\underline{s}^-$ there is $\eta>0$ such that for $m=1,2$ and for all $\varepsilon>0$ small enough there exist $t^\varepsilon_m\in[0,T]$ with $|t^\varepsilon_m-t_m|<\varepsilon$ and $s_m\in(s^-(t^\varepsilon_m)+3\eta, s^+(t^\varepsilon_m)-3\eta)$. (In fact, if $s_m>0$ then take t^ε_m such that $s^+(t^\varepsilon_m)\to\overline{s}^+(t_m)$ and if $s_m<0$ then take t^ε_m such that $s^-(t^\varepsilon_m)\to\underline{s}^-(t_m)$ as $\varepsilon\downarrow 0$.)
On the other hand, for small ε, by continuity of σ at (t_1,t_2) and at (t_2,t_1) (see (71)) and setting $m'=1$ if $m=2$ and viceversa we have $|\sigma(t^\varepsilon_m,t^\varepsilon_{m'})-s_m|<\eta$ because $\sigma(t_m,t_{m'})=s_m$. Thus $\sigma(t^\varepsilon_m,t^\varepsilon_{m'})\in\Big(s^-(t^\varepsilon_m),s^+(t^\varepsilon_m)\Big)$ for ε small enough. This holds for $m=1,2$. But this contradicts admissibility of Γ, see e.g. (69). We conclude that $t_1=t_2$, and so Φ is injective on $\hat{M}$. Since Φ is also continuous on this set (because $\hat{M}\subset\bar{M}$), this readily implies that Φ^{-1} is continuous on $\Phi(\hat{M})$. □

5.3 Extension of countably developable mappings

The next proposition shows that the chart Φ_Γ associated with a uniformly admissibile curve in fact extends as a homeomorphism (a diffeomorphism if Γ is smooth) to a domain containing the closure of $M_{s^\pm}$. We will not need this in the present paper, but it will be used in [10] to construct isometric extensions of a given isometric immersion. We begin by proving the following lemma:

Lemma 10 *Let $\pm s^\pm:[0,T]\to\mathbb{R}$ be bounded, lower semicontinuous and bounded from below by a positive constant, and let $\Gamma\in W^{2,\infty}((0,T);\mathbb{R}^2)$ be $s^\pm$-admissible. Then the following are equivalent:*

(i) $\Phi_\Gamma:\overline{M}_{s^\pm}\to\overline{[\Gamma(0,T)]}$ is a homeomorphism.

(ii) The maps $\beta^\pm_\Gamma:[0,T]\to\mathbb{R}^2$ are injective, and $\beta^+_\Gamma([0,T])\cap\beta^-_\Gamma([0,T])=\emptyset$.

Proof. Recall from Proposition 10 that

$$\overline{[\Gamma(0,T)]}=\overline{\Phi_\Gamma(M_{s^\pm})}=\Phi_\Gamma(\overline{M}_{s^\pm}). \tag{84}$$

Since $\beta^\pm_\Gamma(t)=\Phi_\Gamma(s^\pm(t),t)$ and since $(s^\pm(t),t)\in\overline{M}_{s^\pm}$, the implication (i) $\implies$ (ii) is immediate.
Now assume that (ii) is satisfied. Since Γ is $s^\pm$-admissible, Φ_Γ is injective on $M_{s^\pm}$ by Proposition 10. Since $\beta^\pm_\Gamma$ are injective on $[0,T]$ and the sets $\beta^\pm_\Gamma([0,T])$ are disjoint, we have

$$\overline{[\Gamma(t)]}\cap\overline{[\Gamma(t')]}=\emptyset \text{ for all } t,t'\in[0,T] \text{ with } t\neq t'.$$

Thus Φ_Γ is injective on $\overline{M}_{s^\pm}$. Since it is also continuous on this set, one readily deduces that Φ_Γ^{-1} exists and is continuous on (84). □

Proposition 11 *Let* $\pm s^{\pm} \in C^0([0,T])$ *be bounded from below by a positive constant and let* $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$ *be uniformly* $s^{\pm}$*-admissible. Assume, moreover, that the maps* $\beta_\Gamma^{\pm} : [0,T] \to \mathbb{R}^2$ *are injective and that* $\beta_\Gamma^-([0,T]) \cap \beta_\Gamma^+([0,T]) = \emptyset$*. For* $\delta > 0$ *define* $\tilde{s}_\delta^{\pm} : \mathbb{R} \to \mathbb{R}$ *by*

$$\pm\tilde{s}_\delta^{\pm}(t) := \begin{cases} \pm s^{\pm}(t) + \delta & \text{if } t \in [0,T], \\ \pm s^{\pm}(0) + \delta & \text{if } t < 0, \\ \pm s^{\pm}(T) + \delta & \text{if } t > T, \end{cases} \tag{85}$$

and set

$$\tilde{M}_\delta := \bigcup_{t \in (-\delta, T+\delta)} \left(\tilde{s}_\delta^-(t), \tilde{s}_\delta^+(t)\right) \times \{t\}. \tag{86}$$

Denote by $\tilde{\Gamma}$ *the curve obtained by extending* Γ *with curvature* $\kappa = 0$ *to all of* $\mathbb{R}$*. Then there is* $\delta > 0$ *such that the following hold:*

(i) $\Phi_{\tilde{\Gamma}}(\tilde{M}_\delta)$ *is open and contains* $\overline{\Phi_\Gamma(M_{s^\pm})}$*.*

(ii) $\tilde{\Gamma}$ *is uniformly* $\tilde{s}_\delta^{\pm}$*-admissible on* $[-\delta, T+\delta]$*.*

(iii) $\Phi_{\tilde{\Gamma}}$ *is (globally) Bilipschitz on the domain* $\tilde{M}_\delta$*.*

(iv) If, in addition, $\Gamma \in C^\infty((0,T);\mathbb{R}^2)$*, then* $\Phi_{\tilde{\Gamma}}$ *is a* C^∞*-diffeomorphism on* $\tilde{M}_\delta \setminus (\mathbb{R} \times \{0,T\})$*.*

Remark. The continuity hypothesis on $s^{\pm}$ cannot be dropped; otherwise it may happen that $\overline{M}_{s^\pm}$ is not contained in $\tilde{M}_\delta$.

Proof. We omit the indices Γ and $\tilde{\Gamma}$. Lemma 10 implies that Φ^{-1} is uniformly continuous on $\Phi(M)$, because it is continuous on the compact set $\overline{\Phi(M)}$. Hence the set

$$N_\delta(y) := \{(s,t) \in \tilde{M}_\delta : \Phi(s,t) = y\}$$

satisfies

$$\operatorname{diam} N_\delta(y) \leq 4\delta + \omega(2C_1\delta) \text{ for all } y \in \Phi(\tilde{M}_\delta) \tag{87}$$

for all $\delta > 0$. Here ω is a modulus of continuity for Φ^{-1} on $\Phi(M)$ and C_1 is the Lipschitz constant of Φ on $\tilde{M}_1$. To prove (87), let $(s_i, t_i) \in \tilde{M}_\delta$, $i = 1, 2$, with $\Phi(s_1, t_1) = \Phi(s_2, t_2) =: y$. By definition of $\tilde{M}_\delta$ there are $(s_i', t_i') \in B_{2\delta}(s_i, t_i) \cap M$. So $|\Phi(s_i', t_i') - \Phi(s_i, t_i)| \leq 2C_1\delta$. Thus (87) follows from $|(s_1,t_1) - (s_2,t_2)| \leq 4\delta + |(s_1', t_1') - (s_2', t_2')| \leq 4\delta + \omega\Big(|\Phi(s_1', t_1') - \Phi(s_2', t_2')|\Big)$.

Since $\tilde{\Gamma}$ is uniformly $s^{\pm}$-admissible on $[0,T]$, by construction it is still uniformly *locally* $\tilde{s}_\delta^{\pm}$-admissible on $\mathbb{R}$ for some $\delta \in (0,1)$. Hence we can apply Lemma 9 to $\tilde{\Gamma}|_{[-\delta, T+\delta]}$ to obtain $\delta_1 > 0$ (independent of δ) such that

$$\Phi(s_1, t_1) = \Phi(s_2, t_2) \implies |(s_1, t_1) - (s_2, t_2)| > \delta_1$$

whenever $(s_i, t_i) \in \tilde{M}_\delta$ with $(s_1, t_1) \neq (s_2, t_2)$. Thus, by choosing δ small enough, from (87) we conclude that $\# N_\delta(y) = 1$ for all $y \in \tilde{M}_\delta$, i.e. Φ is injective on $\tilde{M}_\delta$. Hence $\tilde{\Gamma}$ is $\tilde{s}_\delta^{\pm}$-admissible, hence uniformly $\tilde{s}_\delta^{\pm}$-admissible, on $[-\delta, T+\delta]$. So by Proposition 10 the set $\Phi(\tilde{M}_\delta)$ is open and Φ is Bilipschitz on $\tilde{M}_\delta$. Moreover, we have

$$\overline{\Phi(M)} = \Phi(\bar{M}) \subset \Phi(\tilde{M}_\delta)$$

because clearly $\overline{M} \subset \tilde{M}_\delta$; observe that this last inclusion would be false in general if $s^{\pm}$ were not continuous.

To prove the last part of the proposition, notice that

$$\Phi \in C^\infty(\mathbb{R}^2 \setminus (\mathbb{R} \times \{0,T\}))$$

because $\tilde{\Gamma} \in C^\infty(\mathbb{R} \setminus \{0,T\})$ by the extra hypothesis. Moreover, Φ is injective with $\det \nabla\Phi < 0$ on $\bar{M}_\delta$. □

The injectivity hypothesis on $\beta^\pm_\Gamma$ made in Proposition 11 can be weakened, because it almost follows from uniform admissibility:

Remark 4 *Let $\pm s^\pm : [0,T] \to (0,\infty)$ be continuous, and let $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$ be uniformly $s^\pm$-admissible. Then we have: If $\beta^\pm_\Gamma(0,T)$ are singletons or Jordan arcs, then $\beta^\pm$ are injective on $[0,T]$.*

Proof. We omit the index Γ. Notice that $\beta^\pm$ are continuous because so are $s^\pm$. We claim that $\beta^\pm$ are injective on $[0,T]$. In fact, Φ is locally injective on $\bar{M}_{s^\pm}$ by Lemma 9. So

$$\beta^\pm = \Phi(s^\pm, \cdot)$$

are locally injective on $[0,T]$. Combining this with the fact hat $\beta^\pm(0,T)$ are Jordan arcs implies that $\beta^\pm$ are injective on $(0,T)$. To see this let $\alpha : (0,1) \to \beta^+(0,T)$ be a homeomorphism. Then $\alpha^{-1} \circ \beta$ is locally injective, i.e. locally strictly monotone, i.e. strictly monotone.
So $\beta^\pm$ are injective on $[0,T]$ since $\beta^\pm(T) \notin \beta^\pm([0,T))$ by the definition of a Jordan arc. □

For curves lying in a domain S we note the following criteria under which the hypotheses of Remark 4 are satisfied.

Remark 5 *Let $S \subset \mathbb{R}^2$ be a continuous domain and let $* \in \{-,+\}$. If $s^*_\Gamma \in C^0([0,T])$ then there is $k \in \{0,...,N_S\}$ such that $\beta^*_\Gamma([0,T]) \subset \partial_k S$. In this case, the following are equivalent*

- *$\beta^*_\Gamma([0,T])$ is not a closed curve.*
- *$\beta^*_\Gamma([0,T]) \neq \partial_k S$.*
- *$\beta^*_\Gamma((0,T))$ is a singleton or a Jordan arc.*

Proof. If s^*_Γ is continuous then so is β^*_Γ, so the first part of the claim follows because different components of ∂S have positive distance from each other. Clearly, $\beta^*_\Gamma([0,T])$ is strictly contained in $\partial_k S$ if and only if it is not a closed curve. And if this is the case, then it is a connected subset of a subarc of the Jordan curve $\partial_k S$. Hence it is a singleton or a Jordan arc. The converse is clear. □

5.4 Regularity of the directed distance

5.4.1 Lower semicontinuity.

A simple but important property of ν^S is its lower semicontinuity. It is an immediate consequence of Lemma 2.

Lemma 11 *Let $S \subset \mathbb{R}^2$ be a bounded domain. Then ν^S is lower semicontinuous on $S \times (\mathbb{R}^2 \setminus \{0\})$.*

Proof. Let $(x_n, \mu_n) \to (x, \mu)$ in $S \times (\mathbb{R}^2 \setminus \{0\})$. We may assume without loss of generality that $\mu, \mu_n \in \mathbb{S}^1$. Lemma 2(ii) implies that there exist $\varepsilon_n \downarrow 0$ such that

$$[x]_\mu \subset B_{\varepsilon_n}([x_n]_{\mu_n}) \text{ for all } n \in \mathbb{N}.$$

Hence there exist $t_n \in [0,1]$ such that the point $x + \nu^S(x,\mu)\mu$ is contained in the ε_n-ball centered at

$$x_n + \Big[t_n \nu(x_n, \mu_n) - (1-t_n)\nu(x_n, -\mu_n)\Big]\mu_n \in [x_n]_{\mu_n}$$

Since $\varepsilon_n \downarrow 0$, since $x_n \to x$ and since $\mu_n \to \mu$, this readily implies that

$$|\nu(x,\mu) - \Big[t_n \nu(x_n,\mu_n) - (1-t_n)\nu(x_n,-\mu_n)\Big]| \to 0.$$

The term in square brackets does not exceed $\nu(x_n,\mu_n)$ for any $n \in \mathbb{N}$; hence $\nu(x,\mu) \leq \liminf_{n\to\infty} \nu(x_n,\mu_n)$. □

5.4.2 Transversality.

Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $x \in S$ and $\mu \in \mathbb{R}^2 \setminus \{0\}$. We say that $[x]_\mu^S$ intersects ∂S tangentially in $x^* \in \overline{[x]}_\mu \cap \partial S$ if there exists $\alpha : \mathbb{R} \to \mathbb{R}$ Lipschitz, $v \in \mathbb{S}^1$ and $\varepsilon > 0$ such that $x^* \in \operatorname{graph}_v \alpha|_{(-\varepsilon,\varepsilon)} \subset \partial S$ and $\mu \cdot v^\perp \in \partial\alpha(x^* \cdot v)\ \mu \cdot v$. Here,

$$\partial\alpha(\xi) := \text{convex hull of } \{\lim_{n\to\infty} \alpha'(\xi_n) : \xi_n \to \xi \text{ and } \xi_n \notin Q_\alpha\}$$

denotes Clarke's generalized gradient, see [3]. The set Q_α is an arbitrary $\mathcal{L}^1$-null set that contains the non-Lebesgue points of the distributional derivative of α, so by Rademacher's Theorem the classical derivative $\alpha'(\xi_n)$ exists for all $\xi_n \notin Q_\alpha$. We say that $[x]_\mu$ intersects ∂S transversally at its endpoint x^* if it does not intersect ∂S tangentially at x^*.

Definition 5 *A pair $(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is said to be transversal if $[x]_\mu$ intersects ∂S transversally in $x + \nu(x,\mu)\mu$.*

Observe that if S is convex then every pair $(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\})$ is transversal. The following lemma shows why transversality is an important notion.

Lemma 12 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then the set*

$$\{(x,\mu) \in S \times (\mathbb{R}^2 \setminus \{0\}) : (x,\mu) \textit{ is transversal } \}$$

is open, and ν^S is locally Lipschitz on this set.

Proof. Observe that there exists a Lipschitz function α from a neighbourhood of zero in $\mathbb{R}$ into $\mathbb{R}$ such that $x \in \operatorname{graph}\alpha \subset \partial S$. It follows from transversality and Lemma 13 that there exists $\varepsilon > 0$ and a Lipschitz function

$$\tilde{\nu} : B_\varepsilon(x,\mu) \to (0,\infty)$$

such that

$$y^+ := y + \tilde{\nu}(y,\lambda)\lambda \in \operatorname{graph}\alpha \subset \partial S$$

for all $(y,\lambda) \in B_\varepsilon(x,\mu)$. In particular, y^+ lies in the same connected component $\partial_k S$ of ∂S as $x^+ := x + \nu^S(x,\mu)\mu$. Observe that Lemma 13 also implies that $[y]_\lambda$ intersects ∂S transversally in y^+.
The closed segment $[x,x^+]$ is contained in $\overline{[x]}_\mu \setminus \{x^-\}$, where x^- is the other endpoint of $[x]_\mu$. Thus $[x,x^+] \cap \partial S = \{x^+\} \subset \partial_k S$. Hence by compactness $[x,x^+]$ has positive distance from $\partial_j S$ for $j \neq k$. Thus also $[y,y^+]$ has positive distance from $\partial_j S$ for $j \neq k$, provided ε is small enough. Hence $[y,y^+) \subset [y]_\lambda$ by maximality of $[y]_\lambda$ and because $[y,y^+)$ is connected, because it does not intersect ∂S and because it contains y. Thus y^+ is an endpoint of $[y]_\lambda$. We conclude that

$$\tilde{\nu}(y,\lambda) = \nu^S(y,\lambda) \text{ for all } (y,\lambda) \in B_\varepsilon(x,\mu).$$

□

In the proof of Lemma 12 we used the following result:

Lemma 13 *Let $\alpha : \mathbb{R} \to \mathbb{R}$ be Lipschitz (resp. C^k), let $x, \mu \in \mathbb{R}^2$, $\tilde{\nu}_0 \in \mathbb{R}$ be such that $x + \tilde{\nu}_0\mu \in \operatorname{graph}\alpha$. Denoting the coordinates by subindices, the following holds: If*

$$\mu_2 \notin \partial\alpha\big(x_1 + \tilde{\nu}_0\mu_1\big)\mu_1$$

then there is $\varepsilon > 0$ and a unique mapping $\tilde{\nu} : B_\varepsilon(x, \mu) \to \mathbb{R}$ such that

$$y + \tilde{\nu}(y, \lambda)\lambda \in \operatorname{graph}\alpha \text{ for all } (y, \lambda) \in B_\varepsilon(x, \mu),$$

and

$$\lambda_2 \notin \partial\alpha(y_1 + \tilde{\nu}(y, \lambda)\lambda_1)\lambda_1.$$

Moreover, $\tilde{\nu}$ is Lipschitz (resp. C^k) on $B_\varepsilon(x, \mu)$.

Proof. Set

$$F(y, \lambda, \rho) = x_2 + \rho\lambda_2 - \alpha(x_1 + \rho\lambda_1)$$

and

$$G(y, \lambda, \rho) = \Big(y, \lambda, F(y, \lambda, \rho)\Big).$$

Then G is Lipschitz because so is α, and

$$\det \nabla G(y, \lambda, \rho) = \partial_\rho F(y, \lambda, \rho)$$

wherever it exists. Thus if

$$\lambda_2 \notin \lambda_1 \partial\alpha(y_1 + \rho\lambda_1)$$

then the generalized gradient $\partial G(y, \lambda, \rho)$ only contains matrices with nonzero determinant. By the hypothesis, this condition is satisfied at $(x, \mu, \tilde{\nu}_0)$, so by the Lipschitz Inverse Function Theorem [3] there exists a neighbourhood U of $(x, \mu, \tilde{\nu}_0)$ on which G has a Lipschitz inverse $\hat{G}$. The claim follows by setting $\tilde{\nu}(y, \lambda) = \hat{G}(y, \lambda, 0)$. The corresponding result with C^k follows from the classical implicit function theorem. □

5.5 Admissible, locally admissible and transversal curves on a domain

In this section we consider curves $\Gamma : [0, T] \to S$, which lie inside a bounded Lipschitz domain S. Such a curve Γ is said to be (S-)transversal on $J \subset [0, T]$ if $(\Gamma(t), \pm N(t))$ are transversal for every $t \in J$. If Γ is transversal on $[0, T]$ then $s_\Gamma^\pm$ are Lipschitz on $(0, T)$ by Lemma 12.

Proposition 12 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $\Gamma \in W^{2,\infty}([0, T]; S)$. Then the functions $\pm s_\Gamma^\pm : [0, T] \to \mathbb{R}$ are bounded, lower semicontinuous and bounded from below by a positive constant. Moreover, the following hold:*

(i) If Γ is S-admissible then Γ is locally S-admissible.

(ii) If Γ is S-admissible and transversal on $[0, T]$ (so $s_\Gamma^\pm \in C^0([0, T])$) then there are $i, j \in \{0, ..., N_S\}$ such that $\Gamma([0, T]) \subset A_{ij}$. Moreover,

$$[\Gamma(0, T)] = \Phi_\Gamma(M_{s_\Gamma^\pm}) = \mathcal{C}\Big(S \setminus ([\Gamma(0)] \cup [\Gamma(T)]); \Gamma(0, T)\Big). \tag{88}$$

In particular, $S \cap \partial[\Gamma(0, T)] = [\Gamma(0)] \cup [\Gamma(T)]$.

Remark. Figure 4 (right) shows that in (ii) one cannot drop the hypothesis $s_\Gamma^\pm \in C^0$.

Proof. Boundedness of $|s_\Gamma^\pm|$ follows from boundedness of Γ and S, and boundedness from below follows from compactness of $\Gamma([0,T])$ and of ∂S. Lower semicontinuity of $\pm s_\Gamma^\pm$ follows from that of ν^S (cf. Lemma 11) and from continuity of Γ and N. Part (i) follows from Proposition 10 (ii) by taking $s^\pm := s_\Gamma^\pm$. By what we have just shown, these functions indeed satisfy the hypotheses of Proposition 10.

For the proof of (ii) notice that by continuity of $s_\Gamma^\pm$ and by (80) we have

$$\partial M_{s_\Gamma^\pm} = \bigcup_{t\in[0,T]} \Big(\{s_\Gamma^-(t), s_\Gamma^+(t)\} \times \{t\}\Big) \cup \bigcup_{t\in\{0,T\}} \Big((s_\Gamma^-(t), s_\Gamma^+(t)) \times \{t\}\Big). \tag{89}$$

Thus

$$\Phi_\Gamma(\partial M_{s_\Gamma^\pm}) = [\Gamma(0)] \cup \beta_\Gamma^+([0,T]) \cup [\Gamma(T)] \cup \beta_\Gamma^-([0,T]). \tag{90}$$

The set

$$S' := \mathcal{C}\big(S \setminus ([\Gamma(0)] \cup [\Gamma(T)]); \Gamma(0,T)\big)$$

is well-defined because $\Gamma(0,T)$ is connected and by S-admissibility of Γ on $[0,T]$ it does not intersect $[\Gamma(0)] \cup [\Gamma(T)] \cup \partial S$. Now S' is connected and

$$S' \cap (\partial S \cup [\Gamma(0)] \cup [\Gamma(T)]) = \emptyset.$$

But by (local) admissibility Proposition 10 implies that

$$\partial\Phi_\Gamma(M_{s_\Gamma^\pm}) \subset \Phi_\Gamma(\partial M_{s_\Gamma^\pm}).$$

And by (90) this is contained in $[\Gamma(0)] \cup [\Gamma(T)] \cup \partial S$. So $S' \cap \partial\Phi_\Gamma(M_{s_\Gamma^\pm}) = \emptyset$. Since $\Gamma(\frac{T}{2}) \in S' \cap \Phi_\Gamma(M_{s_\Gamma^\pm})$, connectedness of S' implies that $S' \subset \Phi_\Gamma(M_{s_\Gamma^\pm})$. But $\Phi_\Gamma(M_{s_\Gamma^\pm}) \cap \partial S' = \emptyset$ because $\partial S' \subset [\Gamma(0)] \cup [\Gamma(T)] \cup \partial S$ and $\Phi_\Gamma(M_{s_\Gamma^\pm}) \cap ([\Gamma(0)] \cup [\Gamma(T)]) = \emptyset$ by admissibility. So connectedness of $\Phi_\Gamma(M_{s_\Gamma^\pm})$ implies $\Phi_\Gamma(M_{s_\Gamma^\pm}) \subset S'$. The inclusion $\Gamma([0,T]) \subset A_{ij}$ is immediate from continuity of the $s_\Gamma^\pm$ because different components of ∂S have positive distance from each other.

□

The next proposition addresses the question whether there exists a suitable converse to the implication 'admissible $\implies$ locally admissible' in Proposition 12. It will not be needed in the present paper, nor in [10].

Proposition 13 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $\Gamma \in W^{2,\infty}([0,T]; S)$ be transversal on $[0,T]$. Then (i) and (ii) below are equivalent:*

(i) Γ is locally admissible on $(0,T)$ and

$$[\Gamma(0,T)] \cap \Big([\Gamma(0)] \cup [\Gamma(T)]\Big) = \emptyset. \tag{91}$$

(ii) Γ is admissible on $[0,T]$.

Lemma 14 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $\Gamma \in W^{2,\infty}([0,T]; S)$ be locally admissible and transversal on $[0,T]$, and assume that (91) is satisfied. Then $[\Gamma(0,T)]$ is simply connected.*

Proof. It is clear that $\beta_\Gamma^\pm(t)$ and $[\Gamma(t)]$ are contained in the closure of $[\Gamma(0,T)]$ for all $t \in [0,T]$. Since $\beta_\Gamma^\pm([0,T]) \subset \partial S$, we even have

$$\beta_\Gamma^\pm([0,T]) \subset \partial[\Gamma(0,T)].$$

Hence (90) and (91) imply:

$$\begin{aligned}\Phi_\Gamma(\partial M_{s_\Gamma^\pm}) &= \beta_\Gamma^+([0,T]) \cup \beta_\Gamma^-([0,T]) \cup [\Gamma(0)] \cup [\Gamma(T)] \\ &\subset \partial[\Gamma(0,T)] = \partial\Phi_\Gamma(M_{s_\Gamma^\pm});\end{aligned}$$

observe that (90) is satisfied because Γ is transversal. On the other hand, since Γ is locally $s_\Gamma^\pm$-admissible, Proposition 10 implies that $\Phi_\Gamma(M_{s_\Gamma^\pm})$ is connected and open, and that $\partial\Phi_\Gamma(M_{s_\Gamma^\pm}) \subset \Phi_\Gamma(\partial M_{s_\Gamma^\pm})$. We conclude that

$$\partial\Phi_\Gamma(M_{s_\Gamma^\pm}) = \Phi_\Gamma(\partial M_{s_\Gamma^\pm}).$$

Hence $\partial\Phi_\Gamma(M_{s_\Gamma^\pm})$ is connected because $\partial M_{s_\Gamma^\pm}$ is connected and because Φ_Γ is continuous on $\bar{M}_{s_\Gamma^\pm}$. The connectedness of $\partial M_{s_\Gamma^\pm}$ follows e.g. from VI.4.3 in [17] because $M_{s_\Gamma^\pm}$ is a simply connected domain: it is open by Proposition 10, and it is clearly simply connected. By VI.4.3 in [17] we conclude that $[\Gamma(0,T)] = \Phi_\Gamma(M_{s_\Gamma^\pm})$ is a simply connected domain. □

Proofof Proposition 13. Assume that Γ is admissible on $[0,T]$. Then, in particular, (91) is satisfied. And Proposition 12 implies that Γ is locally admissible. Hence (i) follows from (ii).

Now assume that (i) is satisfied. Let

$$t_1 = \sup\{t \in (0,T] : \Gamma \text{ is admissible on } [0,t]\}. \tag{92}$$

Lemma 2.2 in [9] implies that the set on the right-hand side is nonempty. Hence $t_1 > 0$. By continuity of $s_\Gamma^\pm$, Γ and N it is easy to see that the supremum in (92) is attained, i.e.

$$\Gamma \text{ is admissible on } [0,t_1]. \tag{93}$$

Suppose that we had $t_1 < T$. Lemma 2.2 in [9] implies that there is $\varepsilon > 0$ such that

$$\Gamma \text{ is admissible on } [t_1, t_1+\varepsilon]. \tag{94}$$

Lemma 14 implies that $[\Gamma(0,T)]$ is simply connected. Hence the set

$$[\Gamma(0,T)] \setminus [\Gamma(t_1)] \tag{95}$$

consists of precisely two connected components. By (93) we have

$$[\Gamma(0,t_1)] \cap [\Gamma(t_1)] = \emptyset.$$

Hence, by connectedness of $[\Gamma(0,t_1)]$, the set $[\Gamma(0,t_1)]$ is contained in one of the components of (95); the same is true for $[\Gamma(t_1,t_1+\varepsilon)]$ by (94). Since $\Gamma'(t_1)$ is perpendicular to $[\Gamma(t_1)]$, we conclude that $[\Gamma(0,t_1)]$ and $[\Gamma(t_1,t_1+\varepsilon)]$ must lie in different connected components of (95). In particular,

$$[\Gamma(0,t_1)] \cap [\Gamma(t_1,t_1+\varepsilon)] = \emptyset.$$

By (94, 93) this implies that Γ is admissible on $[0,t_1+\varepsilon]$, contradicting the definition of t_1. We conclude that $t_1 = T$. □

One can also prove a result similar to Proposition 13 in the abstract setting considered earlier (and without the tranversality hypothesis), i.e. the counterpart of Proposition 10 (ii):

Proposition 14 *Let* $*s^*: [0,T] \to \mathbb{R}$ *be bounded, lower semicontinuous and uniformly bounded from below by a positive constant. Let* $\Gamma \in W^{2,\infty}((0,T);\mathbb{R}^2)$ *be uniformly locally* $s^\pm$*-admissible and assume that*

$$\Phi_\Gamma(\partial M_{s^\pm}) \subset \partial\Phi_\Gamma(M_{s^\pm}). \tag{96}$$

Then

$$\Phi_\Gamma(\partial M_{s^\pm}) = \partial\Phi_\Gamma(M_{s^\pm}),$$

and Γ *is uniformly* $s^\pm$*-admissible.*

Remarks.

(i) Figure 4 (right) shows that condition (96) cannot be omitted.

(ii) In the proof we will show that the hypothesis (96) is satisfied if $\Phi_\Gamma(\partial M_{s^\pm})$ is a closed Jordan curve.

Proof. As usual we omit some indices. Since Γ is locally $s^\pm$-admissible, $\Phi(M)$ is connected and open by Proposition 10, and $\partial\Phi(M) \subset \Phi(\partial M)$. Combining this with the hypothesis gives $\partial\Phi(M) = \Phi(\partial M)$. Hence $\partial\Phi(M)$ is connected by continuity of Φ on $\bar{M}$ and by connectedness of ∂M (the latter follows from VI.4.3 in [17] because M is a simply connected domain: it is open by Proposition 10, and it is clearly simply connected). By VI.4.3 in [17] this implies that $\Phi(M)$ is a simply connected domain.
Since $\Gamma \in W^{2,\infty}$ there is $\varepsilon_1 > 0$ such that, for all t, $\Gamma([t-\varepsilon_1, t+\varepsilon_1])$ intersects $[\Gamma(t)]$ only in the point $\Gamma(t)$. For all $t \in (0,T)$ we have $[\Gamma(t)] \subset \Phi(M)$ by definition. Moreover, the endpoints of $[\Gamma(t)]$ are contained in $\Phi(\partial M)$. So by the hypothesis they are contained in $\partial\Phi(M)$. (This is crucial.) Hence by Theorem V.11.7 in [17] the set $\Phi(M)\setminus[\Gamma(t)]$ consists of exactly two connected components. Since $\Gamma(t) \in \Phi(M)$, there is $r > 0$ such that $B_r(\Gamma(t)) \subset \Phi(M)$. So $B_r(\Gamma(t)) \setminus [\Gamma(t)]$ consists of exactly two components, namely the half-balls $B^* := \{x \in B_r(x) : *(x - \Gamma(t)) \cdot \Gamma'(t) > 0\}$. As usual (see e.g. the proof of Lemma 5), the B^+ and B^- are contained in different components of $\Phi(M)\setminus[\Gamma(t)]$. We denote by S_t^* the component of $\Phi(M)\setminus[\Gamma(t)]$ that contains B^*. We conclude that

$$\Gamma\Big([t-\varepsilon_1, t)\Big) \subset S_t^- \text{ for all } t \in (\varepsilon_1, T], \tag{97}$$

$$\Gamma\Big((t, t+\varepsilon_1]\Big) \subset S_t^+ \text{ for all } t \in [0, T-\varepsilon_1). \tag{98}$$

(We set $S_0^+ := \Phi(M)$ and $S_T^- := \emptyset$.) In fact, the sets on the left-hand sides of (97, 98) are connected by continuity. And $\partial S_t^\pm \subset \partial\Phi(M) \cup [\Gamma(t)]$, while $\Gamma([t-\varepsilon_1, t+\varepsilon_1]\setminus\{t\})$ does not intersect $[\Gamma(t)] \cup \partial\Phi(M)$ by definition of ε_1 and since $\Gamma(0,T) \subset \Phi(M)$.
By uniform local admissibility there is $\eta > 0$ with $1 - s^\pm\kappa \geq \eta$. Then Lemma 9 implies (e.g. taking $\delta = \frac{\eta}{2}$) that for all $t \in [0,T)$ there exists $\delta_1(t) > 0$ such that

$$[\Gamma(s)] \cap [\Gamma(t)] = \emptyset \text{ for all } s \in [t - \delta_1(t), t + \delta_1(t)]. \tag{99}$$

We may assume without loss of generality that $\sup_{t\in[0,T]} \delta_1(t) < \varepsilon_1$. By (98), (99) and connectedness of $[\Gamma(s)]$ we conclude that

$$\Big[\Gamma((t-\varepsilon, t])\Big] \subset S_{t-\varepsilon}^+ \text{ and } \Big[\Gamma((t, t+\varepsilon])\Big] \subset S_t^+ \text{ for all } \varepsilon \in (0, \delta_1(t)). \tag{100}$$

The first inclusion follows from (98) with $t - \varepsilon$ instead of t. (Notice that ε_1 does not depend on t.) Next note that $\Gamma(t+\varepsilon) \in S_t^+ \cap \partial S_{t+\varepsilon}^+$ by (100), so by openness $S_t^+ \cap S_{t+\varepsilon}^+ \neq \emptyset$. On the other hand, $\Gamma(t) \in S_{t+\varepsilon}^-$ by (97). Now $[\Gamma(t)]$ is connected and by (99) it does not intersect $\partial S_{t+\varepsilon}^+ \subset [\Gamma(t+\varepsilon)] \cup \partial\Phi(M)$ because $\Phi(M)$ is open. We conclude that $[\Gamma(t)]$ does not intersect $S^+_{t+\varepsilon}$. Thus $S^+_{t+\varepsilon} \cap \partial S^+_t = \emptyset$. By connectedness we conclude that $S^+_{t+\varepsilon} \subset S^+_t$. By a similar argument one proves that $S^+_t \subset S^+_{t-\varepsilon}$. Summarizing,

$$S^+_{t+\varepsilon} \subset S^+_t \subset S^+_{t-\varepsilon} \text{ for all } \varepsilon \in (0, \delta_1(t)). \tag{101}$$

We claim that, for all $t' \in [0, T)$,

$$S^+_t \subset S^+_{t'} \text{ for all } t \in (t', T]. \tag{102}$$

In fact, let $t_1 := \sup\{t'' \geq t' : S^+_t \subset S^+_{t'} \text{ for all } t \in [t', t'']\}$. By (101) we have $S^+_{t_1} \subset S^+_{t_1-\varepsilon} \subset S_{t'}$ for some $\varepsilon > 0$. So the supremum is attained. And if $t_1 < T$ then we would obtain a contradiction, since by (101) we have $S^+_{t_1+\varepsilon} \subset S^+_{t_1} \subset S^+_{t'}$ for all $\varepsilon > 0$ small enough. This proves (102).
Now let $t, t' \in [0, T]$ with $t' < t$. Then $[\Gamma(t)] \subset S^+_{t''}$ for some $t'' \in (t', t)$ by the first inclusion in (100). Thus $[\Gamma(t)] \subset S^+_{t'}$ by (102). So $[\Gamma(t')] \cap [\Gamma(t)] = \emptyset$. This proves that Γ is $s^\pm$-admissible. Since it is also uniformly locally $s^\pm$-admissible, we conclude that it is uniformly $s^\pm$-admissible.
Let us finally prove Remark (ii) to Proposition 14:

Claim. If Γ is locally $s^\pm$-admissible and $\Phi_\Gamma(\partial M_{s^\pm})$ is a closed Jordan curve then $\Phi_\Gamma(M_{s^\pm}) = U_b\Big(\Phi_\Gamma(\partial M_{s^\pm})\Big)$. In particular, $\partial\Phi_\Gamma(M_{s^\pm}) = \Phi_\Gamma(\partial M_{s^\pm})$.

To prove the claim set $\alpha := \Phi(\partial M)$ (we omit the index Γ). Then α is a Jordan curve and $\partial\Phi(M) \subset \alpha$ by Proposition 10. Thus $U_b(\alpha) \cap \partial\Phi(M) = U_\infty(\alpha) \cap \partial\Phi(M) = \emptyset$. Since $U_\infty(\alpha)$ is connected and $\Phi(M)$ is bounded, we conclude $\Phi(M) \cap U_\infty(\alpha) = \emptyset$. Hence by openness $\Phi(M) \subset U_b(\alpha)$. In particular, $U_b(\alpha)$ intersects $\Phi(M)$, whence by connectedness $U_b(\alpha) \subset \Phi(M)$ because $U_b(\alpha) \cap \partial\Phi(M) = \emptyset$. This proves the claim. □

5.6 Regularity of the directed distance along admissible curves

In this section we are interested in the regularity of the functions

$$s^\pm_\Gamma(t) = \pm\nu^S(\Gamma(t), \pm N(t))$$

for a given curve $\Gamma \in W^{2,\infty}([0,T]; S)$. Of course the regularity of ν^S established in Section 5.4 carries over to $s^\pm_\Gamma$. However, the Lipschitz continuity for transversal curves Γ is not enough for our purposes, nor is the global lower semicontinuity of ν^S. In this section, we will show that if Γ is admissible (but possibly non-transversal) then $s^\pm_\Gamma$ are of bounded variation. This, of course, has important consequences about the regularity of the set $[\Gamma(0,T)]$; more precisely about the relative boundary $S \cap \partial[\Gamma(0,T)]$.

5.6.1 Admissible curves sweep ∂S monotonically.

That $s^\pm_\Gamma$ are of bounded variation will be an immediate consequence of the following lemma. It shows that the mappings

$$\beta^\pm_\Gamma(t) = \Gamma(t) + s^\pm_\Gamma(t)N(t)$$

sweep ∂S in a monotone fashion. This is clearly false in general if Γ is not admissible. The proof of Lemma 15 is slightly complicated, but the result is rather intuitive, at least on simply connected domains. For domains which are not simply connected, the lemma also shows that $\beta^\pm_\Gamma$ jump only a controlled number of times from one component of ∂S to the another. This is true for a purely topological reason. As the arguments in the proof are topological, we do not require Lipschitz regularity of ∂S. Recall that $N_S + 1$ is the number of connected components of ∂S.

Lemma 15 *Let $S \subset \mathbb{R}^2$ be a continuous domain and let $\Gamma \in W^{2,\infty}([0,T];S)$ be S-admissible. Then, for all $k \in \{0,...,N_S\}$ and $* \in \{-,+\}$ with*

$$(\beta_\Gamma^*)^{-1}(\partial_k S) \neq \emptyset$$

there is $P_k^* \in \{1,...,N_S+1\}$ and there are disjoint intervals $J_1^{k,*},...,J_{P_k^*}^{k,*} \subset [0,T]$ such that

$$(\beta_\Gamma^*)^{-1}(\partial_k S) = J_1^{k,*} \cup ... \cup J_{P_k^*}^{k,*}.$$

If $t' \in \{\sup J_i^{k,*}, \inf J_i^{k,*}\}$ is contained in $J_i^{k,*}$, then $[\Gamma(t')]$ intersects $\partial_k S$ tangentially in $\beta_\Gamma^*(t')$. Moreover, each restriction $\beta_\Gamma^*|_{J_i^{k,*}}$ is monotone in the following sense:
If $\alpha : \mathbb{S}^1 \to \partial_k S$ is a homeomorphism then there is $\theta \in (-2\pi, 0]$ and a monotone lifting $\phi_i^{k,*} : J_i^{k,*} \to [\theta, \theta + 2\pi]$ for $\alpha^{-1} \circ \beta_\Gamma^*$, i.e.

$$\beta_\Gamma^*(t) = \alpha(e^{i\phi_i^{k,*}(t)}) \text{ for all } t \in J_i^{k,*}.$$

The lifting $\phi_i^{k,*}$ is non-degenerate in the following sense: If $\beta_\Gamma^*|_{J_i^{k,*}}$ is constant then $\phi_i^{k,*}$ is constant as well (i.e. it does not attain both values θ and $\theta + 2\pi$).

Remarks.

(i) Each interval $J_i^{k,*}$ can be open, closed, or half-closed.

(ii) The bound $P_k^* \leq N_S + 1$ is clearly sharp.

(iii) If S is Lipschitz and $\overline{[\Gamma(t)]}$ intersects ∂S transversally in $\beta_\Gamma^*(t)$ then β_Γ^* is Lipschitz in a small neighbourhood J of t by Proposition 15. Then one can roughly argue as follows: Since β_Γ^* is continuous on J, its image is contained in one Bilipschitz chart α of $\partial_k S$. Set $\rho := \alpha^{-1}(\beta_\Gamma^*)$. Then $(\beta_\Gamma^*)' = \alpha'(\rho)\rho'$. Since $(\beta_\Gamma^*)' = (1 - s_\Gamma^*\kappa)\Gamma' + (s_\Gamma^*)'N$ one finds $(\alpha'(\rho) \cdot \Gamma')\rho' = 1 - s^*\kappa$. The factor $\alpha'(\rho) \cdot \Gamma' = |\alpha'(\rho)|\hat{\nu}(\beta_\Gamma^*) \cdot N$ is nonnegative (for $k \neq 0$) because $N \cdot \hat{\nu}(\beta_\Gamma^*) \geq 0$. (Here $\hat{\nu}(x)$ denotes the outer unit normal to S at x, if it exists.) Thus by (local) admissibility $\rho' \geq 0$, which means that β_Γ^* is (locally) monotone.
Such a local argument breaks down at points t where $\overline{[\Gamma(t)]}$ intersect ∂S tangentially; as seen in Figure 4, the set of such t need not be small. In general, the image of a small interval under β_Γ^* is not contained in a small subarc of ∂S: It need not even be contained in a single component of ∂S.

Proof. We omit some indices. Let Γ be as in the hypothesis and let us consider the case $* = +$; the case $* = -$ is similar. We define $[\Gamma(t)]_+ := \{\Gamma(t) + sN(t) : s \in (0, s^+(t))\}$. For $(t_0, t_1) \subset (0,T)$ we define

$$M_{t_0,t_1}^+ := \{(s,t) : t \in (t_0,t_1) \text{ and } s \in (0, s^+(t))\}. \quad (103)$$

As in Proposition 10 each set M_{t_0,t_1}^+ is open by lower semicontinuity of s^+. So $[\Gamma(t_0,t_1)]_+ := \Phi(M_{t_0,t_1}^+)$ is open because Φ is open by Proposition 10 and admissibility of Γ. Define

$$Z_{t_0,t_1} := \Gamma([t_0,t_1]) \cup [\Gamma(t_0)]_+ \cup [\Gamma(t_1)]_+.$$

Assume that $\beta^+(t_0)$ and $\beta^+(t_1)$ lie in the same component $\partial_k S$ of ∂S. Set

$$\tilde{S}_k := \begin{cases} U_b(\partial_0 S) & \text{if } k = 0 \\ U_\infty(\partial_k S) & \text{if } k \neq 0. \end{cases}$$

Notice that $Z_{t_0,t_1} \subset \tilde{S}_k$. Two cases can occur: If $\beta^+(t_0) = \beta^+(t_1)$, then $Z_{t_0,t_1} \cup \{\beta^+(t_0)\}$ is a closed Jordan curve. If $\beta^+(t_0) \neq \beta^+(t_1)$, then Z_{t_0,t_1} is a Jordan arc with both endpoints $\beta^+(t_0), \beta^+(t_1) \in \partial_k S$. In either case the set

$$S^+_{t_0,t_1} := \mathcal{C}(\tilde{S}_k \setminus Z_{t_0,t_1}; [\Gamma(t_0,t_1)]_+) \tag{104}$$

is well defined. In fact, $[\Gamma(t_0,t_1)]_+ \subset \tilde{S}_k$ is connected (by continuity of Φ and connectedness of $M^+_{t_0,t_1}$), and it does not intersect $Z_{t_0,t_1} \cup \partial\tilde{S}_k$ (by admissibility of Γ). If $\beta^+(t_0) \neq \beta^+(t_1)$ then by Lemma 16 (i) the set $S^+_{t_0,t_1}$ is a Jordan domain and there is a connected component L_{t_0,t_1} of $\partial_k S \setminus \{\beta^+(t_0), \beta^+(t_1)\}$ such that

$$\partial S^+_{t_0,t_1} = L_{t_0,t_1} \cup \bar{Z}_{t_0,t_1}. \tag{105}$$

We claim that if $(t_0,t_1) \subset (0,T)$ then

$$\overline{[\Gamma(t)]}_+ \cap S^+_{t_0,t_1} = \emptyset \text{ for all } t \in [0,T] \setminus (t_0,t_1). \tag{106}$$

To prove (106) note first that since $S^+_{t_0,t_1}$ is open, it suffices to prove it with $[\Gamma(t)]_+$ instead of $\overline{[\Gamma(t)]}_+$. Moreover, $([\Gamma(t_0)]_+ \cup [\Gamma(t_1)]_+) \cap S^+_{t_0,t_1} = \emptyset$ because $[\Gamma(t_0)]_+ \cup [\Gamma(t_1)]_+ \subset Z_{t_0,t_1}$ does not intersect $S^+_{t_0,t_1}$ by its definition (104). Thus it remains to prove that $[\Gamma(t)]_+ \cap S^+_{t_0,t_1} = \emptyset$ for all $t \in [0,T] \setminus [t_0,t_1]$. So without loss of generality we may assume that $t_0 > 0$ and $t_1 < T$.
Set $\delta := \frac{1}{2} \inf \mathrm{dist}_{\partial S}\Big(\Gamma([0,T])\Big)$ and set $\Gamma_\delta(t) := \Gamma(t) + \delta N(t)$. So $\Gamma_\delta(t_0,t_1) \subset [\Gamma(t_0,t_1)]_+ \subset S^+_{t_0,t_1}$. For small ε, the curve $\Gamma_\delta(t_1-\varepsilon, t_1+\varepsilon)$ intersects $\partial S^+_{t_0,t_1}$ in exactly one point, namely in $\Gamma_\delta(t_1)$. (To prove this notice that by closedness of $\Gamma([t_0,t_1]) \cup \overline{[\Gamma(t_0)]}_+$ and since $\Gamma_\delta(t_1) \in S$, there is $\varepsilon > 0$ such that $B_\varepsilon(\Gamma_\delta(t_1)) \cap \partial S^+_{t_0,t_1} \subset [\Gamma(t_1)]_+$. And $\Gamma_\delta(t_1-\varepsilon, t_1+\varepsilon) \cap [\Gamma(t_1)] = \{\Gamma_\delta(t_1)\}$ for small ε because $\Gamma'_\delta(t_1)$ is perpendicular to $[\Gamma(t_1)]$.) By a curve index argument we conclude that, for all $\varepsilon > 0$ small enough, $\Gamma_\delta(t_1+\varepsilon)$ and $\Gamma_\delta(t_1-\varepsilon)$ lie in different connected components of $\mathbb{R}^2 \setminus (\bar{Z}_{t_0,t_1} \cup L_{t_0,t_1})$ (if $\beta^+(t_0) \neq \beta^+(t_1)$) resp. of $\mathbb{R}^2 \setminus (Z_{t_0,t_1} \cup \beta^+(t_0))$ (if $\beta^+(t_0) = \beta^+(t_1)$). Since $\Gamma_\delta(t_1-\varepsilon) \in [\Gamma(t_0,t_1)]_+ \subset S^+_{t_0,t_1}$, we conclude that $\Gamma_\delta(t_1+\varepsilon) \notin S^+_{t_0,t_1}$ for small ε. But $[\Gamma((t_1,T])]_+ \subset S$ is connected and does not intersect $\partial S^+_{t_0,t_1} \subset \partial S \cup Z_{t_0,t_1}$ (by admissibility), while it contains $\Gamma_\delta(t_1+\varepsilon)$ for small $\varepsilon > 0$. Thus $[\Gamma((t_1,T])]_+ \cap S^+_{t_0,t_1} = \emptyset$. An analogous argument at t_0 concludes the proof of (106).
We claim that

$$S^+_{t_0,t_1} \subset S^+_{t'_0,t'_1} \text{ if } (t_0,t_1) \subset (t'_0,t'_1). \tag{107}$$

In fact, $S^+_{t_0,t_1}$ clearly intersects $S^+_{t'_0,t'_1}$ because $[\Gamma(t_0,t_1)]_+ \subset S^+_{t_0,t_1} \cap S^+_{t'_0,t'_1}$. But by (106) the curve $Z_{t'_0,t'_1}$ does not intersect $S^+_{t_0,t_1}$, since $\Gamma(t) \in \overline{[\Gamma(t)]}_+$ for all t. Thus $S^+_{t_0,t_1} \cap \partial S^+_{t'_0,t'_1} = \emptyset$, and (107) follows from connectedness of $S^+_{t_0,t_1}$.

Claim #1. If $t_2 \in (t_0,t_1) \subset (0,T)$ are such that $\beta^+(t_0), \beta^+(t_1) \in \partial_k S$ and $\beta^+(t_2) \in \partial_i S$ with $i \neq k$, then $(\beta^+)^{-1}(\partial_i S) \subset (t_0,t_1)$.

In fact, since $[\Gamma(t_0,t_1)]_+ \subset S^+_{t_0,t_1}$, we have $\beta^+(t_0,t_1) \subset \Phi(\bar{M}^+_{t_0,t_1}) \subset \overline{\Phi(M^+_{t_0,t_1})} = \overline{[\Gamma(t_0,t_1)]}_+ \subset \bar{S}^+_{t_0,t_1}$. So $\beta^+(t_2) \in \bar{S}^+_{t_0,t_1}$ and so $\bar{S}^+_{t_0,t_1} \cap \partial_i S \neq \emptyset$. But $\partial_i S$ does not intersect $\partial S^+_{t_0,t_1} \subset S \cup \partial_k S$. So $S^+_{t_0,t_1} \cap \partial_i S \neq \emptyset$. Since $\partial_i S$ is connected, we conclude that $\partial_i S \subset S^+_{t_0,t_1}$. But by (106), the closures of $[\Gamma([0,t_0])]_+$ and of $[\Gamma([t_1,T])]_+$ do not intersect $S^+_{t_0,t_1}$. So $\Big(\beta^+([0,t_0]) \cup \beta^+([t_1,T])\Big) \cap \partial_i S = \emptyset$. This proves Claim #1.

We claim that $(\beta^+)^{-1}(\partial_k S)$ consists of at most $N_S + 1$ connected components for each $k = 0, ..., N_S$. In fact, define $F : [0,T] \to \{0, ..., N_S\}$ by setting $F(t) := k$ iff $\beta^+(t) \in \partial_k S$. Suppose the claim were false for some k. Then there would exist $0 \leq t_1 < t'_1 < ... < t_{N_S+1} < t'_{N_S+1} < t_{N_S+2} \leq T$ such that $F(t'_i) \neq k$ for all

$i = 1, ..., N_S + 1$ and $F(t_i) = k$ for all $i = 1, ..., N_S + 2$. By Claim #1 we therefore have $F^{-1}(F(t'_i)) \subset (t_i, t_{i+1})$ for all $i = 1, ..., N_S + 1$. So the sets $F^{-1}(F(t'_i))$ are pairwise disjoint. Hence $F(t'_1), ..., F(t'_{N_S+1})$ must be pairwise different. Since F takes values in $\{0, ..., N_S\}$ and since $F(t'_i) \neq k$, this is impossible. This proves the claim. Using it, we obtain $J_i^{k,+}$ as in the statement of the lemma: Each $J_i^{k,+}$ is a connected component of $(\beta^+)^{-1}(\partial_k S)$.
Now assume that $J_i^{k,+}$ contains $t' \in \{\inf J_i^{k,+}, \sup J_i^{k,+}\}$. If $\overline{[\Gamma(t')]}$ intersected $\partial_k S$ transversally at $\beta^+(t')$ then by Lemma 12 and continuity of Γ and N the segment $\overline{[\Gamma(t)]}$ would intersect $\partial_k S$ in $\beta^+(t)$ for t near t', contradicting extremality of t'.

Now fix $k \in \{0, ..., N_S\}$ and $i \in \{1, ..., P_k^+\}$. Let us prove monotonicity of $\beta^+|_{J_i^{k,+}}$. If $J_i^{k,+}$ is a singleton (we did not exclude this possibility) then there is nothing to prove. So let us assume that $J_i^{k,+}$ is nondegenerate.

Claim #2. Let $t_0, t_1 \in J_i^{k,+}$ with $t_0 < t_1$. If $\beta^+(t_0) = \beta^+(t_1)$ then either $\partial_k S \setminus \{\beta^+(t_0)\} \subset S^+_{t_0,t_1}$ or $\Big(\partial_k S \setminus \{\beta^+(t_0)\}\Big) \cap \bar{S}^+_{t_0,t_1} = \emptyset$.

To prove this, notice that $\partial S_{t_0,t_1} = Z_{t_0,t_1} \cup \{\beta^+(t_0)\}$ is a closed Jordan curve that does not intersect $\partial_k S \setminus \{\beta^+(t_0)\}$. So if $\partial_k S \setminus \{\beta^+(t_0)\}$ intersects the closure of $S^+_{t_0,t_1}$ then it must intersect $S^+_{t_0,t_1}$ itself. Since $\partial_k S \setminus \{\beta^+(t_0)\}$ is connected, we conclude that it must then be contained in $S^+_{t_0,t_1}$. This proves Claim #2.

Let $\alpha : \mathbb{S}^1 \to \partial_k S$ be a homeomorphism and let $[T_0, T_1] \subset J_i^{k,+}$. Assume that $\beta^+(T_0) \neq \beta^+(T_1)$, so L_{T_0,T_1} (see (105)) is well defined. For given $x_0 \in \mathbb{S}^1$ let $\theta \in (-2\pi, 0]$ be the solution of $e^{i\theta} = x_0$. Denote by $\arg_{x_0} : \mathbb{S}^1 \to [\theta, \theta + 2\pi)$ the unique branch of the argument function with this range which is continuous on $\mathbb{S}^1 \setminus \{x_0\}$.
Fix $y_0 \in \partial_k S \setminus \bar{L}_{T_0,T_1}$ and let $\theta \in (-2\pi, 0]$ solve $e^{i\theta} = \alpha^{-1}(y_0)$. Define $\phi : [T_0, T_1] \to [\theta, \theta + 2\pi)$ by

$$\phi(t) := \arg_{\alpha^{-1}(y_0)} \Big(\alpha^{-1}(\beta^+(t))\Big) \text{ for all } t \in [T_0, T_1]. \tag{108}$$

Then $\alpha(e^{i\phi}) = \beta^+$ on $[T_0, T_1]$, so ϕ is a lifting for $\alpha^{-1}(\beta^+)$.
Let $[t_0, t_1] \subset [T_0, T_1]$ be such that $\beta^+(t_0) \neq \beta^+(t_1)$. We claim that then

$$L_{t_0,t_1} = \alpha(e^{i(\phi(t_0)\phi(t_1))}) := \{\alpha(e^{i\varphi}) : \varphi \in (\phi(t_0)\phi(t_1))\}. \tag{109}$$

Here and below we use the notation $(ab) := (\min\{a, b\}, \max\{a, b\})$, and we use a similar notation for closed intervals.
Let us prove (109). Clearly $(\phi(t_0)\phi(t_1))$ is an open subinterval of $[\theta, \theta + 2\pi)$ with endpoints $\phi(t_0), \phi(t_1)$. Hence $|\phi(t_0) - \phi(t_1)| < 2\pi$, so $\varphi \mapsto e^{i\varphi}$ is injective on $[\phi(t_0)\phi(t_1)]$. Hence $e^{i(\phi(t_0)\phi(t_1))}$ is an open subarc of $\mathbb{S}^1$ with endpoints $e^{i\phi(t_0)}$ and $e^{i\phi(t_1)}$. Thus by (108) and since α is a homeomorphism, the right-hand side of (109) is an open subarc of $\partial_k S$ with endpoints $\beta^+(t_0)$ and $\beta^+(t_1)$. Moreover, $\theta \notin [\phi(t_0)\phi(t_1)]$. So the closure of the right-hand side of (109) does not contain $y_0 = \alpha(e^{i\theta})$. On the other hand, L_{t_0,t_1} is also a subarc of $\partial_k S$ with endpoints $\beta^+(t_0)$ and $\beta^+(t_1)$, see the line above (105). Since $\bar{L}_{t_0,t_1} = \bar{S}_{t_0,t_1} \cap \partial_k S \subset \bar{S}_{T_0,T_1} \cap \partial_k S = \bar{L}_{T_0,T_1}$ by (107), we also have $y_0 \notin \bar{L}_{t_0,t_1}$. Summarizing, for both sides of (109) we know: It agrees with a connected component of $\partial_k S \setminus \{\beta^+(t_0), \beta^+(t_1)\}$, and y_0 is contained in the other component. Since $\partial_k S \setminus \{\beta^+(t_0), \beta^+(t_1)\}$ consists of precisely two components, (109) follows.

Claim #3. Suppose $[t_0, t_1] \subset [T_0, T_1]$. If $\beta^+(t_0) = \beta^+(t_1)$ then $\beta^+([t_0, t_1]) = \{\beta^+(t_0)\}$. If $\beta^+(t_0) \neq \beta^+(t_1)$ then $\beta^+([t_0, t_1]) \subset \bar{L}_{t_0,t_1}$.

In fact, consider first the case $\beta^+(t_0) = \beta^+(t_1)$. Since $\beta^+(T_0) \neq \beta^+(T_1)$ we have $\beta^+(t_0) \neq \beta^+(T_0)$ or $\beta^+(t_1) \neq \beta^+(T_1)$. Suppose that $\beta^+(t_0) \neq \beta^+(T_0)$; the other case is similar. By (106) we have $\overline{[\Gamma(T_0)]}_+ \cap S^+_{t_0,t_1} = \emptyset$. Hence $\beta^+(T_0) \notin$

$S^+_{t_0,t_1}$. But on the other hand, $\beta^+(T_0) \in \partial_k S \setminus \{\beta^+(t_0)\}$. Thus Claim #2 implies that $\Big(\partial_k S \setminus \{\beta^+(t_0)\}\Big) \cap \bar{S}^+_{t_0,t_1} = \emptyset$. Since $\beta^+([t_0,t_1]) \subset \bar{S}^+_{t_0,t_1} \cap \partial_k S$, this implies $\beta^+([t_0,t_1]) = \{\beta^+(t_0)\}$.
Next consider the case $\beta^+(t_0) \neq \beta^+(t_1)$, so $t_0 < t_1$. Then by definition of $S^+_{t_0,t_1}$ we have $\beta^+([t_0,t_1]) \subset \bar{S}^+_{t_0,t_1} \cap \partial_k S = \bar{L}_{t_0,t_1}$ by (105). This concludes the proof of Claim #3.

Let $[t_0,t_1] \subset [T_0,T_1]$. If $\beta^+(t_0) \neq \beta^+(t_1)$ then by Claim #3 we have $\beta^+([t_0,t_1]) \subset \bar{L}_{t_0,t_1}$. By (109) this implies that $\beta^+([t_0,t_1]) \subset \alpha\Big(e^{i[\phi(t_0)\phi(t_1)]}\Big)$. Applying $\arg_{\alpha^{-1}(y_0)} \circ \alpha^{-1}$ to both sides gives

$$\phi([t_0,t_1]) \subset [\phi(t_0)\phi(t_1)]. \tag{110}$$

If $\beta^+(t_0) = \beta^+(t_1)$ then Claim #3 implies that $\beta^+([t_0,t_1]) \subset \{\beta^+(t_0)\}$, so (110) holds as well. Thus (110) holds for all closed subintervals $[t_0,t_1] \subset [T_0,T_1]$. From this one readily deduces that $\phi : [T_0,T_1] \to [\theta, \theta + 2\pi)$ is a monotone function.

Let us now consider the case $\beta^+(T_0) = \beta^+(T_1) =: \beta^+_0$. If $\beta^+([T_0,T_1]) = \{\beta^+_0\}$ then there is nothing to prove. Let us therefore assume that there is $T_2 \in (T_0,T_1)$ such that $\beta^+(T_2) \neq \beta^+_0$. We claim that

$$L_{T_0,T_2} \cap L_{T_2,T_1} = \emptyset. \tag{111}$$

In fact, applying Theorem 11.8 in [17] to the Jordan domain $\tilde{S}_k$ and the Jordan arc Z_{T_0,T_2} we find that the component S' of $\tilde{S}_k \setminus Z_{T_0,T_2}$ which is not $S^+_{T_0,T_2}$ satisfies $\partial S' = Z_{T_0,T_2} \cup \bar{L}'_{T_0,T_2}$. Here L'_{T_0,T_2} is the connected component of $\partial_k S \setminus \{\beta^+(T_0), \beta^+(T_2)\}$ that is not L_{T_0,T_2}. By (106) and the usual connectedness argument we have $S^+_{T_2,T_1} \subset S'$. (Indeed, for $t \in (T_2,T_1)$ we have: $[\Gamma(t)]_+ \subset S^+_{T_2,T_2}$ by definition. And $[\Gamma(t)]_+ \subset S'$ by (105, 106) and by definition of S'. Thus $S^+_{T_2,T_1} \cap S' \neq \emptyset$. Moreover, $S^+_{T_2,T_1}$ is connected and does not intersect $\partial S' \subset \partial_k S \cup Z_{T_0,T_2}$.) Thus $\bar{L}_{T_2,T_1} \subset \partial_k S \cap \bar{S}' = \bar{L}'_{T_0,T_2}$. This does not intersect L_{T_0,T_2}, so (111) is proven.
Applying the first part of this proof to $[T_0,T_2]$ and to $[T_2,T_1]$, we conclude that there exist $\theta_0, \theta_1 \in \mathbb{R}$ and monotone functions $\phi_0 : [T_0,T_2] \to [\theta_0, \theta_0 + 2\pi)$ and $\phi_1 : [T_2,T_1] \to [\theta_1, \theta_1 + 2\pi)$ such that $\beta^+ = \alpha(e^{i\phi_0})$ on $[T_0,T_2]$ and $\beta^+ = \alpha(e^{i\phi_1})$ on $[T_2,T_1]$. In particular, $\alpha(e^{i\phi_0(T_2)}) = \alpha(e^{i\phi_1(T_2)})$. So there is $\tilde{n} \in \mathbb{Z}$ such that, setting $\tilde{\phi}_1 := \phi_1 + 2\pi\tilde{n}$ we have $\phi_0(T_2) = \tilde{\phi}_1(T_2)$. Define $\phi : [T_0,T_1] \to \mathbb{R}$ setting by

$$\phi(t) := \begin{cases} \phi_0(t) & \text{if } t \in [T_0,T_2] \\ \tilde{\phi}_1(t) & \text{if } t \in [T_2,T_1]. \end{cases} \tag{112}$$

By (111) and by (109) we conclude that $(\phi(T_0)\phi(T_2)) \cap (\phi(T_2)\phi(T_1)) = \emptyset$. Since $\phi|_{[T_0,T_2]}$ and $\phi|_{[T_2,T_1]}$ are monotone, this implies that ϕ is monotone.
It remains to check that it takes values in an interval of length 2π. To be explicit suppose that ϕ is nondecreasing; the other case is similar. We claim that

$$\phi([T_0,T_1]) \subset [\phi(T_0), \phi(T_1)] = [\phi(T_0), \phi(T_0) + 2\pi]. \tag{113}$$

In fact, the first inclusion follows from monotonicity. It therefore suffices to prove that $\phi(T_1) = \phi(T_0) + 2\pi$. Since ϕ is nondecreasing and nonconstant and since $\beta^+(T_1) = \beta^+(T_0)$, there is an integer $n \geq 1$ such that $\beta^+(T_1) = 2\pi n + \phi(T_0)$. But since $|\phi(T_2) - \phi(T_0)| < 2\pi$ and $|\phi(T_1) - \phi(T_2)| < 2\pi$, we cannot have $n > 1$. This concludes the proof of (113).

Summarizing, we have shown that for all closed intervals $[T_0,T_1] \subset J^{k,+}_i$ there exists a monotone lifting $\phi : [T_0,T_1] \to [\theta, \theta + 2\pi]$ for $\alpha^{-1}(\beta^+)$, where $\theta := \min\{\phi(T_0), \phi(T_1)\}$. Set $J^- := \inf J^{k,+}_i$ and $J^+ := \sup J^{k,+}_i$ and let $\phi_n : [J^- + \frac{1}{n}, J^+ - \frac{1}{n}] \to [\theta_n, \theta_n + 2\pi]$ be a monotone lifting for $\alpha^{-1}(\beta^+)$ with $\theta_n := \min\{\phi_n(J^- +$

$\frac{1}{n}), \phi_n(J^+ - \frac{1}{n})\}$. After possibly adding an integer multiple of 2π to each ϕ_n we may assume that $\phi_n = \phi_m$ on $[J^- + \frac{1}{m}, J^+ - \frac{1}{m}]$ if $m \le n$. Set $\phi_n(t) := \phi_n(J^- + \frac{1}{n})$ for $t < J^- + \frac{1}{n}$ and $\phi_n(t) := \phi_n(J^+ - \frac{1}{n})$ for $t > J^+ - \frac{1}{n}$. Then the ϕ_n are monotone on $J_i^{k,+}$ and they converge pointwise to a function $\phi : J_i^{k,+} \to \mathbb{R}$, which is therefore monotone as well. And it is clearly a lifting for $\alpha^{-1}(\beta^+)$. Moreover, $\phi(J_i^{k,+})$ has length not greater than 2π because the same is true for all $\phi_n(J_i^{k,+})$. This concludes the proof. □

5.6.2 Regularity of $s_\Gamma^\pm$.

Corollary 2 *If $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $\Gamma \in W^{2,\infty}([0,T]; S)$ is admissible, then $s_\Gamma^\pm \in BV(0,T)$.*

Proof. We use the notation from the proof of Lemma 15. Since S is a Lipschitz domain, there exists a homeomorphism $\alpha : \mathbb{S}^1 \to \partial_k S$ which is Lipschitz. By the proof of Lemma 15, for all $m = 1, ..., P_k^+$ there are $\theta_m \in \mathbb{R}$ and there are monotone liftings $\phi_m : J_m^{k,+} \to [\theta_m, \theta_m + 2\pi]$ such that $\beta^+(t) = \alpha(e^{i\phi_m(t)})$ for all $t \in J_m^{k,+}$. By monotonicity and boundedness of ϕ_m, the functions $e^{i\phi_m}$ have bounded variation. Hence Theorem 3.99 in [1] implies that also $\beta^+|_{J_m^{k,+}}$ has bounded variation for all $k = 0, ..., N_S$ and for all $i = 1, ..., P_k^+$. Since

$$[0,T] = \bigcup_{k=0}^{N_S} \bigcup_{i=1}^{P_k^+} J_m^{k,+},$$

we conclude that $\beta^+ \in BV((0,T); \mathbb{R}^2)$. Hence by [1] Proposition 3.2 also $s^+ = \beta^+ \cdot N - \Gamma \cdot N \in BV(0,T)$ since N is Lipschitz. □

Corollary 3 *Let $S \subset \mathbb{R}^2$ be a continuous domain and let $\Gamma \in W^{2,\infty}([0,T]; S)$ be S-admissible. Then there exists a partition $0 =: T_0 < T_1 < ... < T_M := T$ such that for all $i = 1, ..., M$ and $* \in \{-, +\}$ there is $k_i^* \in \{0, ..., N_S\}$ such that*

$$\beta_\Gamma^*(T_{i-1}, T_i) \subset \partial_{k_i^*} S. \tag{114}$$

Moreover, the closure of $\beta_\Gamma^*(T_{i-1}, T_i)$ does not contain $\partial_{k_i^*} S$.

Proof. We omit the index Γ. For each $k = 0, ..., N_S$ fix a homeomorphism $\alpha_k : \mathbb{S}^1 \to \partial_k S$. Using the notation and conclusion of Lemma 15, there clearly exists a partition $0 =: T_0 < ... < T_M := T$ such that for all $i = 1, ..., M$ there are k_i^+, m_i^+ such that $(T_{i-1}, T_i) = \operatorname{int} J_{m_i^+}^{k_i^+,+}$. And there is a monotone lifting $\phi_{m_i^+}^{k_i^+,+}$ for $\alpha_{k_i^+}^{-1}(\beta^+)$ on (T_{i-1}, T_i).

Now fix $i \in \{1, ..., M\}$. If $|\phi_{m_i^+}^{k_i^+,+}(T_i) - \phi_{m_i^+}^{k_i^+,+}(T_{i-1})| < 2\pi$ then we do nothing. If $|\phi_{m_i^+}^{k_i^+,+}(T_i) - \phi_{m_i^+}^{k_i^+,+}(T_{i-1})| = 2\pi$ then by the non-degeneracy part of Remark (i) to Lemma 15 there is $T_i' \in (T_{i-1}, T_i)$ such that $\beta^+(T_i') \neq \beta^+(T_{i-1}) = \beta^+(T_i)$. Thus $|\phi_{m_i^+}^{k_i^+,+}(T_i) - \phi_{m_i^+}^{k_i^+,+}(T_i')| < 2\pi$ and $|\phi_{m_i^+}^{k_i^+,+}(T_{i-1}) - \phi_{m_i^+}^{k_i^+,+}(T_i')| < 2\pi$.

Including the T_i' thus obtained into the partition and relabelling we obtain a partition $0 = T_0 < T_1 < .. < T_M = T$ (for some possibly different M than at the beginning) such that, for all $i = 1, ..., M$, the set $\beta_\Gamma^+(T_{i-1}, T_i)$ is contained in one connected component $\partial_{k_i^+} S$ of S and such that

$$|\phi_{m_i^*}^{k_i^*,*}(T_{i-1}) - \phi_{m_i^*}^{k_i^*,*}(T_i)| < 2\pi \text{ for all } i = 1, ..., M \tag{115}$$

and for $* = +$. Finally, we apply the same argument with $-$ instead of $+$ to each restriction $\Gamma|_{(T_{i-1},T_i)}$, $i = 1, ..., M$ to obtain a refined partition which, after relabelling, satisfies (114, 115) simultaneously for both $* = +, -$ (and again with a different M).
To prove the final statement, notice that if $\operatorname{clos} \beta^*(T_{i-1}, T_i) = \partial_{k_i^*} S$ then necessarily $|\phi_{m_i^*}^{k_i^*,*}(T_{i-1}) - \phi_{m_i^*}^{k_i^*,*}(T_i)| = 2\pi$. This contradicts (115). □

For $\Gamma \in W^{2,\infty}([0, T]; S)$ we introduce the sets

$$D_\Gamma^\pm = \left\{t \in [0, T] : [\Gamma(t)]_{N(t)} \text{ intersects } \partial S \text{ tangentially at } \Gamma(t) + s^\pm(t)N(t).\right\}.$$

Observe that Γ is transversal on $J \subset [0, T]$ if $(D_\Gamma^+ \cup D_\Gamma^-) \cap J = \emptyset$.

Proposition 15 *Let $* \in \{+, -\}$, let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $\Gamma \in W^{2,\infty}([0, T]; S)$. Then $*s_\Gamma^*$ is lower semicontinuous. Moreover:*

(i) D_Γ^* is closed, and for all $t' \in [0, T]\backslash D_\Gamma^*$ there is a neighbourhood of $\big(\Gamma(t'), *N(t')\big)$ on which ν is Lipschitz, and $[x]_\mu$ intersects ∂S transversally in $x + *\nu(x, *\mu)\mu$ for all (x, μ) in this neighbourhood. In particular, $[\Gamma(t)]$ intersects ∂S transversally in $\beta_\Gamma^*(t)$ for all t in a neighbourhood of t' and $s_\Gamma^* \in W_{loc}^{1,\infty}\big((0, T) \setminus D_\Gamma^*\big)$.

(ii) If Γ is S-admissible then $s_\Gamma^* \in BV(0, T)$. The set $A_{s_\Gamma^*}$ of discontinuities of s_Γ^* is countable, and at all $t \in A_{s_\Gamma^*}$ the right and left limits of s_Γ^* exist. Moreover, $A_{s_\Gamma^*} \subset D_\Gamma^*$.

(iii) Assume that S satisfies condition $(*)$ from [10], i.e. there is a closed subset $\Sigma \subset \partial S$ with $\mathcal{H}^1(\Sigma) = 0$ such that the outer unit normal $\hat{\nu}$ to S exists and is continuous on $\partial S\backslash\Sigma$. If Γ is S-admissible, then the sets $F_\Gamma^* := D_\Gamma^* \cup (\beta_\Gamma^*)^{-1}(\Sigma)$ are closed, and $s_\Gamma^* \kappa = 1$ almost everywhere on F_Γ^*. Moreover, ν is C^1 in a neighbourhood of $\big(\Gamma(t'), *N(t')\big)$ whenever $t' \in (0, T) \setminus F_\Gamma^*$.

Remarks.

(i) If S is convex then $D_\Gamma^\pm = \emptyset$ and Γ is always transversal on $[0, T]$.

(ii) The inclusion $A_{s_\Gamma^*} \subset D_\Gamma^*$ can be strict, see e.g. Figure 4 (middle).

Proof. Lower semicontinuity of $*s_\Gamma^*$ was proven in Proposition 12. In what follows, we omit the subindex Γ, and for definiteness we take $* = +$; the proofs for $* = -$ are similar.
To prove (i) let $t' \in [0, T] \setminus D^+$. Then by Lemma 12 there exists a neighbourhood of $(\Gamma(t'), N(t'))$ on which ν is Lipschitz, and $\overline{[x]}_\mu$ intersects ∂S transversally in $x^+ = x + \nu(x, \mu)\mu$ for (x, μ) in this neighbourhood. By continuity of (Γ, N) this implies that $\overline{[\Gamma(t)]}$ intersects ∂S transversally in $\beta^+(t)$ for all t near t'. Thus $[0, T] \setminus D^+$ is relatively open in $[0, T]$, so D^+ is closed. Since Γ and N are Lipschitz, $s^+ = \nu(\Gamma, N)$ is therefore Lipschitz in a neighbourhood of t'. In particular, s^+ is continuous at t'. This shows that D^+ contains A_{s^+}.

Let us prove (ii). By Corollary 2 we know $s^+ \in BV((0, T); \mathbb{R}^2)$. For each $k \in \{0, ..., N_S\}$ with $(\beta^+)^{-1}(\partial_k S) \neq \emptyset$, Proposition 15 yields $P_k^+ \in \mathbb{N}$ and intervals $J_i^{k,+}$ such that $\beta^+(J_i^{k,+}) \subset \partial_k S$ for $i = 1, ..., P_k^+$. Fix k and i. By Proposition 15 there is a nondecreasing bounded function $\phi_{k,i} : \bar{J}_i^{k,+} \to \mathbb{R}$ and there is $\alpha : \mathbb{S}^1 \to \partial_k S$ continuous such that $\beta^+ = \alpha(e^{i\phi_{k,i}})$ on $J_i^{k,+}$. Since $\phi_{k,i}$ is monotone, by the remark preceding Corollary 3.29 in [1] it is a good representative in their terminology. So by Theorem 3.28 in [1] there exists a countable set $A_i^{k,+} \subset \bar{J}_i^{k,+}$ (namely the set of atoms of the distributional derivative of $\phi_{k,i}$) such that $\phi_{k,i}|_{J_i^{k,+}} \in C^0(\bar{J}_i^{k,+} \setminus A_i^{k,+})$. Moreover, by monotonicity $\phi_{k,i}$ has one-sided limits at every point in $\bar{J}_i^{k,+}$. Since

$\beta^+ = \alpha(e^{i\phi_{k,i}})$ on $J_i^{k,+}$ (so $s^+ = \alpha(e^{i\phi_{k,i}}) \cdot N - \Gamma \cdot N$), we deduce that s^+ has one-sided limits at every $t \in \bar{J}_i^{k,+}$ and $J_i^{k,+} \cap A_{s^+} \subset A_i^{k,+}$. So indeed $J_i^{k,+} \cap A_{s^+}$ is countable. Applying this argument for each i, k, we conclude

$$A_{s^+} \subset \bigcup_{k=0}^{N_S} \bigcup_{i=1}^{M_k} \left(\{\inf J_i^{k,+}, \sup J_i^{k,+}\} \cup (A_{s^+} \cap J_i^{k,+}) \right).$$

This is countable, and since the union of the closures of the $J_i^{k,+}$ covers $[0,T]$, one-sided limits of s^+ exist at all $t \in [0,T]$.

To prove (iii) suppose that ∂S satisfies condition $(*)$ from the introduction. Set $\Sigma'' := (\beta^+)^{-1}(\Sigma)$. By Proposition 3.92a in [1] we have

$$(\beta^+)' = 0 \text{ almost everywhere on } \Sigma''. \tag{116}$$

Here and below, the prime denotes the density of the part of the distributional derivative which is absolutely continuous with respect to Lebesgue measure. Since N is Lipschitz with $N' = -\kappa\Gamma'$ and since s^+ is BV, the product rule for BV functions (see Proposition 3.2 (b) in [1]) implies that

$$(\beta^+)' = (1 - s^+\kappa)\Gamma' + (s^+)'N. \tag{117}$$

Comparing (116) and (117) shows that $s^+\kappa = 1$ almost everywhere on Σ''.
We claim that $s^+\kappa = 1$ almost everywhere on $D^+ \setminus \Sigma''$ as well. Since ∂S is covered by finitely many Lipschitz graphs, it suffices to prove that $s^+\kappa = 1$ almost everywhere on $D^+ \setminus (\beta^+)^{-1}(\Sigma \cap G)$ for all Lipschitz graphs $G \subset \partial S$. Fix such G and let k be such that $G \subset \partial_k S$. Lemma 15 implies that $(\beta^+)^{-1}(G) \cap J_i^{k,+}$ is connected for all $i = 1, ..., P_k^+$. This is an easy consequence of the monotonicity of $\beta^+|_{J_i^{k,+}}$. Choose coordinates such that $G = \operatorname{graph} \alpha|_E$ for some Lipschitz function α and some open interval E.
Thus, setting $\beta_j^+ := e_j \cdot \beta^+$, we have $\beta_2^+ = \alpha(\beta_1^+)$ on $(\beta^+)^{-1}(G)$. Applying Theorem 3.99 in [1] on (the interior of) the interval $(\beta^+)^{-1}(G) \cap J_i^{k,+}$ for all $i = 1, ..., P_k^+$, we conclude that

$$(\beta_2^+)' = \alpha'(\beta_1^+)(\beta_1^+)' \text{ almost everywhere on } (\beta^+)^{-1}(G). \tag{118}$$

(As usual, the primes denotes the absolutely continuous part of the distributional derivative.) On the other hand, since Σ is closed with $\mathcal{H}^1(\Sigma) = 0$ and since $\xi \mapsto (\xi, \alpha(\xi))$ is Bilipschitz, the set $\Sigma' := \{\xi : (\xi, \alpha(\xi)) \in \Sigma\}$ is closed and has measure zero. And from the definition of Σ it is easy to deduce that $\alpha \in C^1(E \setminus \Sigma')$. Thus

$$\partial\alpha(\xi) = \{\alpha'(\xi)\} \text{ for all } \xi \in E \setminus \Sigma'. \tag{119}$$

Hence $N(t) \cdot e_2 = \alpha'(\beta_1^+(t))N(t) \cdot e_1$ for all $t \in D^+ \cap (\beta^+)^{-1}(G \setminus \Sigma)$. Together with (118) this implies that $(\beta^+)'$ is parallel to N almost everywhere on $D^+ \cap (\beta^+)^{-1}(G \setminus \Sigma)$. So $s^+\kappa = 1$ almost everywhere on $D^+ \cap (\beta^+)^{-1}(G \setminus \Sigma)$ by (117). Thus indeed $s^+\kappa = 1$ almost everywhere on F^+.
Moreover, if $t' \in [0,T] \setminus F^+$ then since $t' \notin D^+$ we know that β^+ is continuous in a neighbourhood of t'. Hence $\operatorname{dist}_\Sigma(\beta^+) > 0$ near t' because $\operatorname{dist}_\Sigma(\beta^+(t')) > 0$ by closedness of Σ. This proves that $[0,T] \setminus F^+$ is relatively open. To prove C^1-regularity of ν one argues as in (i) and in Lemma 12, but now one uses the C^1-version of Lemma 13. □

5.7 Integral curves

Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S;\mathbb{R}^P)$ be countably S-developable. A curve $\Gamma \in W^{2,\infty}([0,T];S)$ with $\Gamma([0,T]) \subset S \setminus \hat{C}_f$ is called an *f-integral curve* if it satisfies the following conditions:

$$\Gamma'(t) = -\hat{q}_f^{\perp}(\Gamma(t)) \text{ for all } t \in [0,T], \text{ and} \tag{120}$$

$$\Gamma'(t)\cdot\Gamma'(t') > 0 \text{ for all } t, t' \in [0,T]. \tag{121}$$

Here $\hat{q}_f : S \setminus \hat{C}_f \to \mathbb{S}^1$ is some S-ruling for f that extends q_f. Its existence is ensured by Proposition 9. Notice that, since $\Gamma([0,T]) \subset S$, the ruling $\hat{q}_f$ can be chosen Lipschitz in a neighbourhood of $\Gamma([0,T])$ by virtue of Remark 1. Clearly, if the length T of Γ does not exceed $1/\|\kappa\|_{L^\infty(0,T)}$, then (121) is satisfied. When f is the gradient of an isometric immersion $u : S \to \mathbb{R}^3$ then curves satisfying (120) are lines of curvature of u.

Remark 6 *If $x \in \mathbb{R}^2$, $r > 0$ and $f : B_r(x) \to \mathbb{R}^P$ is $B_r(x)$-developable then there is a unique solution $\Gamma \in W^{2,\infty}([-\frac{r}{4}, \frac{r}{4}]; B_r(x))$ of (63) with $\Gamma(0) = x$. It satisfies $B_{\frac{r}{8}}(x) \subset [\Gamma(-\frac{r}{4}, \frac{r}{4})]$. Moreover, $[\Gamma((0, \frac{r}{4}])]$ and $[\Gamma([-\frac{r}{4}, 0))]$ are contained in different connected components of $B_r(x) \setminus [\Gamma(0)]$, and Γ satisfies (121).*

Proof. Clearly $\tilde{f}(y) := f(x + ry)$ is $B_1(0)$-developable on $B_1(0)$. After translating and rescaling we may therefore assume without loss of generality that $x = 0$ and $r = 1$. Proposition 16 implies that Γ is $B_1(0)$-admissible. Since Γ solves (120), we have $|\Gamma'| = 1$, so indeed $\Gamma([-\frac{1}{4}, \frac{1}{4}]) \subset B_1(0)$.
Let us prove $B_{\frac{r}{8}}(x) \subset [\Gamma(-\frac{r}{4}, \frac{r}{4})]$. Since $B_1(0)$ is convex, Γ is transversal on $[-\frac{1}{4}, \frac{1}{4}]$. Thus $[\Gamma(-\frac{1}{4}, \frac{1}{4})] = \mathcal{C}\Big(B_1(0) \setminus ([\Gamma(-\frac{1}{4})] \cup [\Gamma(\frac{1}{4})]); 0\Big)$ by Proposition 12 (iii). By connectedness it is therefore enough to show that $B_{\frac{1}{8}}(0)$ does not intersect $[\Gamma(-\frac{1}{4})] \cup [\Gamma(\frac{1}{4})]$. Since $\Gamma([-\frac{1}{4}, \frac{1}{4}]) \subset \bar{B}_{\frac{1}{4}}(0)$, we have $|s_\Gamma^{\pm}| \geq \frac{3}{4}$ on $[-\frac{1}{4}, \frac{1}{4}]$. Thus $|\kappa| \leq \frac{4}{3}$ almost everywhere on $(-\frac{1}{4}, \frac{1}{4})$ because Γ is locally admissible by Proposition 10 (ii). Hence we can estimate

$$\begin{aligned} |\Gamma(t) + sN(t)| &\geq (\Gamma(t) + sN(t))\cdot\Gamma'(t) \\ &= \int_0^t \Gamma'(\xi)\cdot\Gamma'(t)\, d\xi \geq |t| - \frac{4}{3}t^2 \end{aligned}$$

for all $t \in [-\frac{1}{4}, \frac{1}{4}]$. Thus $|\Gamma(\pm\frac{1}{4}) + sN(\pm\frac{1}{4})| \geq \frac{1}{6}$ for all $s \in \mathbb{R}$. And so $B_{\frac{1}{8}}(0) \cap ([\Gamma(-\frac{1}{4})] \cup [\Gamma(\frac{1}{4})]) = \emptyset$.
To prove the second part of the statement notice that by connectedness of $[\Gamma((0, \frac{1}{4}])]$ and of $[\Gamma([-\frac{1}{4}, 0))]$, it is enough to show that these sets do not intersect $[\Gamma(0)]$ and that they intersect different components of $B_1(0) \setminus [\Gamma(0)]$. But the first fact follows from admissibility and the second one because $\Gamma'(0)$ is perpendicular to $[\Gamma(0)]$. Finally, (121) is satisfied because Γ has length $\frac{1}{2} < \frac{3}{4} \leq \frac{1}{\|\kappa\|_{L^\infty(-\frac{1}{4}, \frac{1}{4})}}$. □

The next proposition collects some properties of f-integral curves. Most importantly, such curves are admissible and the domain $[\Gamma(0,T)]$ which they parametrize enjoys the regularity property (B_f) from Definition 2.

Proposition 16 *Let S be a bounded Lipschitz domain, let $f \in C^0(S;\mathbb{R}^P)$ countably developable and let $\Gamma \in W^{2,\infty}([0,T]; S \setminus \hat{C}_f)$ be an f-integral curve. Then Γ is S-admissible on $[0,T]$ with*

$$[\Gamma(t)] = [\Gamma(t)]^S_{\hat{q}_f(\Gamma(t))} \text{ for all } t \in [0,T],$$

and

$$S \cap \partial[\Gamma(0,T)] \subset S \setminus \hat{C}_f$$

and $[\Gamma(0,T)]$ *satisfies condition* (B_f). *Moreover, the following are true:*

(i) *For all*

$$x \in (S \cap \partial[\Gamma(0,T)]) \setminus ([\Gamma(0)] \cup [\Gamma(T)]),$$

we have either

$$[x] \subset \{\Gamma(t) + sN(t) : s \in (s_\Gamma^+(t), \overline{s}_\Gamma^+(t))\} \text{ for some } t \in A_{s_\Gamma^+} \tag{122}$$

or

$$[x] \subset \{\Gamma(t) + sN(t) : s \in (\underline{s}_\Gamma^-(t), s_\Gamma^-(t))\} \text{ for some } t \in A_{s_\Gamma^-}. \tag{123}$$

Here, $A_{s_\Gamma^\pm}$ *is as in (78) and* $\overline{s}_\Gamma^+$, $\underline{s}_\Gamma^-$ *are as in (77).*

(ii) *We have*

$$f(\Phi_\Gamma) \in C^0(\overline{M}_{s_\Gamma^\pm}; \mathbb{R}^P) \tag{124}$$

and

$$f \in C^0(\Phi_\Gamma(\hat{M}); \mathbb{R}^P), \tag{125}$$

where

$$\hat{M} := \bigcup_{t \in [0,T]} (\underline{s}_\Gamma^-(t), \overline{s}_\Gamma^+(t)) \times \{t\}.$$

Moreover,

$$f(x) = f(\Gamma(t)) \text{ for all } x \in S \cap \{\Gamma(t) + sN(t) : s \in (\underline{s}_\Gamma^-(t), \overline{s}_\Gamma^+)\}. \tag{126}$$

Remarks.

(i) A condition like (121) is needed to ensure admissibility of Γ. In fact, on domains which are not simply connected, a curve which only satisfies (120) could spiral or circle around a hole in S: Take e.g. $S = B_2(0) \setminus \bar{B}_1(0)$ and $f(x) = q_f(x) = \frac{x}{|x|}$.

(ii) The inclusions (122, 123) are strict in general. Consider e.g. $S = \{(x_1, x_2) : x_1 \in (-4\pi, 4\pi); x_2 \in (-5, \cos x_1)\}$ and $\Gamma(t) = (t-2)e_2$ on $[0, \frac{3}{2}]$ with $t = 1 \in A_{s_\Gamma^+} \cap A_{s_\Gamma^-}$.

Proof. From (120) we have $N(t) = \hat{q}_f(\Gamma(t))$, so by the definitions of $s_\Gamma^\pm$ and of $[\Gamma(t)]$, we have $[\Gamma(t)] = [\Gamma(t)]_{N(t)} = [\Gamma(t)]_{\hat{q}_f(\Gamma(t))}$. Since $\hat{q}_f$ is an S-ruling, this implies that $[\Gamma(t_0)] \cap [\Gamma(t_1)] = \emptyset$ whenever $\Gamma(t_1) \notin [\Gamma(t_0)]$. Therefore, to prove admissibility we must only show that $\Gamma([0,T]) \cap [\Gamma(t_0)] = \{\Gamma(t_0)\}$ for all $t_0 \in [0,T]$. But for $t \neq t_0$ by (121) we have $(\Gamma(t) - \Gamma(t_0)) \cdot \Gamma'(t_0) \neq 0$. Hence indeed $\Gamma(t) \notin [\Gamma(t_0)]^{\mathbb{R}^2}$.
Next we prove property (B_f). Let $x \in S \cap \partial[\Gamma(0,T)]$. Then there are $t_n \in [0,T]$ and $s_n \in (s^-(t_n), s^+(t_n))$ such that $\Gamma(t_n) + s_n N(t_n) \to x$. So x is contained in the Hausdorff limit Y of $\overline{[\Gamma(t_n)]_{\hat{q}_f(\Gamma(t_n))}}$ (which exists after possibly passing to a subsequence). So by Lemma 2 (ii) we have $[x] \subset Y$. So $[x] \subset \overline{[\Gamma(0,T)]}$. On the other hand, if $[x]$ intersected $[\Gamma(t)]$ for some t then by developability we would have $x \in [x] = [\Gamma(t)] \subset [\Gamma(0,T)]$. Since $[\Gamma(0,T)]$ is open by admissibility and by Proposition 10, this would contradict $x \in \partial[\Gamma(0,T)]$. Thus $[x] \cap [\Gamma(0,T)] = \emptyset$, so $[x] \subset \partial[\Gamma(0,T)]$. Since $[\Gamma(0,T)] = \Phi_\Gamma(M_{s_\Gamma^\pm})$ is connected by continuity of Φ_Γ, we conclude that $[\Gamma(0,T)]$ satisfies (B_f).
If $x \in S \cap \partial\Phi_\Gamma(M_{s_\Gamma^\pm})$ then $[x] \subset \partial\Phi_\Gamma(M_{s_\Gamma^\pm}) \subset \Phi_\Gamma(\partial M_{s_\Gamma^\pm})$ by condition (B_f) and

by Proposition 10. Thus by (80) and since $\Phi_\Gamma(s_\Gamma^\pm,\cdot)$, $\Phi_\Gamma(\overline{s}_\Gamma^+,\cdot)$ and $\Phi_\Gamma(\underline{s}_\Gamma^-,\cdot)$ take values in ∂S (by closedness of ∂S), we have

$$[x] \subset S \cap \Phi_\Gamma(\partial M_{s_\Gamma^\pm}) \subset \Phi_\Gamma\Big(\bigcup_{t \in A_{s_\Gamma^+}} (s_\Gamma^+(t), \overline{s}_\Gamma^+(t)) \times \{t\} \cup \bigcup_{t \in A_{s_\Gamma^-}} (\underline{s}_\Gamma^-(t), s_\Gamma^-(t)) \times \{t\} \Big),$$

or $x \in [\Gamma(0)] \cup [\Gamma(T)]$. Of course, only one of the two inclusions (122) or (123) can hold since the sets on the right are disjoint by injectivity of Φ_Γ, which follows by admissibility from Proposition 10.

To prove part (ii) observe that (124) is true simply because by definition of Γ

$$f(\Phi_\Gamma(s,t)) = f(\Gamma(t)) \text{ for all } t \in [0,T] \text{ and } s \in (s_\Gamma^-(t), s_\Gamma^+(t)).$$

But Proposition 10 implies that Φ_Γ is injective on $\hat{M}$. Since $\hat{M} \subset \overline{M}_{s_\Gamma^\pm}$, one readily deduces (125) from (124). Formula (126) follows immediately from the previous assertions. □

6 Decomposition of the domain S into subdomains which are compatible with f

The main result of this chapter is Theorem 4, which is the full version of Theorem 2 in the introduction. In order to describe its content, let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S;\mathbb{R}^P)$ be finitely developable. Theorem 4 yields a two-step decomposition of S: The first (rough) step of the decomposition resembles the one used in Theorem 3: It decomposes S into a large subdomain W_0 (containing S_δ^f) and countably many small subdomains $W_1, W_2, ...$ which are located near ∂S and whose closures intersect that of W_0 in a single line segment each. The second (fine) step of the decomposition involves W_0. It is decomposed into *finitely* many subdomains $V_1, ..., V_N$ satisfying the compatibility condition (B_f) from Definition 2. More precisely, each of these subdomains is either one of the finitely many connected components of $\hat{C}_f$ or a domain of the form $[\Gamma(0,T)]$ for an f-integral curve Γ which is 'almost transversal'. The use of such a decomposition is explained after the statement the theorem.

Theorem 4 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain and let $f \in C^0(S;\mathbb{R}^P)$ be finitely S-developable. Then there is $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$ there exists $N \in \mathbb{N}$ and subdomains*

$$V_1, ..., V_N \subset S$$

as well as a (possibly infinite) family of subdomains

$$W_1, W_2, ... \subset S$$

such that the set

$$W_0 := \operatorname{int}\left(\bigcup_{k=1}^N \bar{V}_k\right) \tag{127}$$

is a subdomain of S and such that the following are true:

(i) **Disjoint interiors:** *Whenever $j \neq k$ we have*

$$W_j \cap W_k = \emptyset \ (j,k \in \{0,1,...\}) \text{ and } V_j \cap V_k = \emptyset \ (j,k \in \{1,...,N\}). \tag{128}$$

(ii) **Closures intersect nicely:** *For every $k \geq 1$ there exists $x_k \in S \setminus \hat{C}_f$ such that*

$$S \cap \bar{W}_j \cap \bar{W}_k = \begin{cases} \emptyset & \text{if } 0 < j < k \\ [x_k] & \text{if } 0 = j < k. \end{cases}$$

Moreover, with S^f_δ as in (54),

$$i \neq j \text{ and } x \in \bar{V}_i \cap \bar{V}_j \cap S \implies x \in S^f_\delta \setminus \hat{C}_f \text{ and } [x] \subset \partial V_i \cap \partial V_j \cap S; \quad (129)$$

in particular, $\mathcal{H}^1([x]) \geq \delta$.

(iii) **Covering:** *We have*

$$\hat{C}_f \subset S^f_\delta \subset W_0 \quad (130)$$

and

$$S = W_0 \cup \bigcup_{k \geq 1} (S \cap \bar{W}_k).$$

(iv) **Local properties of the covering:** *More precisely, there is $M \in \mathbb{N}$ with $M \leq N$ such that*

$$V_1, ..., V_M \text{ are the connected components of } \hat{C}_f,$$

and for all $k \geq M+1$ there exist $T_k > 0$ and f-integral curves

$$\Gamma^{(k)} \in W^{2,\infty}([0, T_k]; S \setminus \hat{C}_f) \quad (131)$$

such that $V_k = [\Gamma^{(k)}(0, T_k)]$. Furthermore, if $i < j$ and $S \cap \bar{V}_i \cap \bar{V}_j \neq \emptyset$ then $j \geq M+1$, and

$$S \cap \bar{V}_i \cap \bar{V}_j \subset \bigcup_{t \in \{0, T_j\}} \{\Gamma^{(j)}(t) + sN^{(j)}(t) : s \in (\underline{s}^-_{\Gamma^{(j)}}(t), \overline{s}^+_{\Gamma^{(j)}}(t))\} \quad (132)$$

Remark 7 *(i) The proof shows that each of the curves $\Gamma^{(k)}$ in (131) is 'almost transversal' in the sense that the functions $s^\pm_{\Gamma^{(k)}}$ only have small jumps except at the endpoints 0 and T_k. More precisely,*

$$\sup_{t \in (0, T_k)} |s^+_{\Gamma^{(k)}}(t) - \overline{s}^+_{\Gamma^{(k)}}(t)| < \delta \text{ and } \sup_{t \in (0, T_k)} |s^-_{\Gamma^{(k)}}(t) - \underline{s}^-_{\Gamma^{(k)}}(t)| < \delta. \quad (133)$$

(ii) Despite the previous remark, the covering must allow sets $[\Gamma^{(k)}(0, T_k)]$ with non-transversal $\Gamma^{(k)}$. So, in general,

$$S \cap \partial[\Gamma^{(k)}(0, T_k)] \neq [\Gamma^{(k)}(0)] \cup [\Gamma^{(k)}(T_k)].$$

The set on the left typically consists of infinitely many segments of the form $[x]$.
However, each set V_k satisfies condition (B_f); this follows from Proposition 16 and from Proposition 7. Hence Lemma 5 implies that each V_k has finite perimeter, so (129) implies that $\bar{V}_i \cap \bar{V}_j \cap S$ consists of at most finitely many disjoint segments of the form $[x]$.

(iii) From (128, 130) we deduce that

$$W_k \subset S \setminus \bar{S}^f_\delta \text{ for all } k \geq 1.$$

Since $S \setminus \bar{S}^f_\delta \subset B_\delta(\partial S)$, in view of (128) this implies that

$$\sum_{k \geq 1} \mathcal{L}^2(W_k) \leq C\delta$$

for some constant C depending only on S.

A decomposition as in Theorem 4 is useful for the following reason: If we are able to modify f locally on the nice subdomains V_i, then we can also modify it globally by pasting together finitely many local modifications. The idea is to construct the global modification of $f : S \to \mathbb{R}^P$ in two steps: First glue together the local modifications (e.g. smoothings) $f^{(k)} : S \cap \bar{V}_k \to \mathbb{R}^P$; they must satisfy boundary conditions on the intersections $S \cap \bar{V}_k \cap \bar{V}_j$ in order to yield a well-defined glued mapping; say we impose the condition $f^{(k)} = f$ on these intersections. Theorem 4 shows that these intersections consist of only finitely many (long) segments. Thus after the first step one has obtained a well-defined mapping $F_{W_0} : S \cap \bar{W}_0 \to \mathbb{R}^P$ given by

$$F_{W_0}(x) = f^{(k)}(x) \text{ if } x \in S \cap \bar{V}_k \text{ for some } k = 1, ..., N.$$

In the second step one ignores the initial mapping f and defines the mapping $\tilde{F} : S \to \mathbb{R}^P$ by setting

$$\tilde{F}(x) = \begin{cases} F_{W_0}(x) & \text{if } x \in S \cap \bar{W}_0 \\ F_k(x) & \text{if } x \in S \cap \bar{W}_k \text{ for some } k \geq 1 \end{cases}$$

for some mappings $F_k : S \cap \bar{W}_k \to \mathbb{R}^P$. The mapping $\tilde{F}$ is well-defined if and only if

$$F_k = F_{W_0} \text{ on } [x_k] \text{ for each } k \geq 1;$$

this is because $\bar{W}_k \cap \bar{W}_0 \cap S = [x_k]$. In [10] such a construction is applied to an isometric immersions $u : S \to \mathbb{R}^3$ in order to obtain smooth global approximants of u from smooth local approximants on each V_k.

The following proposition shows that most conclusions of Theorem 4 are true for any disjoint finite covering with sets V_k satisfying condition (B_f). Lemma 5 shows that for every subdomain $V \subset S$ with $S \cap \partial V \subset S \setminus \hat{C}_f$ satisfying condition (B_f), there exists a countable subset Z_V of the relative boundary $S \cap \partial V$ such that

$$S \cap \partial V = \bigcup_{x \in Z_V} [x] \text{ (disjoint union).}$$

And if $x \in Z_V$ and both endpoints of $[x]$ are contained in the same connected component of ∂S (i.e. $x \in A^f_{ii}$ for some $i \in \{0, ..., N_S\}$, where $N_S + 1$ is the number of connected components of ∂S) then $S \setminus [x]$ consists of precisely two connected components S^1_x and S^2_x, determined by the fact that $V \subset S^1_x$. Moreover, one has $S \cap \overline{S}^2_x \cap \overline{V} = [x]$ and $S \cap \bar{S}^2_x \cap \bar{S}^2_y = \emptyset$ if $x, y \in Z_V$ with $x \neq y$. These notations and facts will be used in the rest of the chapter.

Proposition 17 *Let $S \subset \mathbb{R}^2$ be a bounded Lipschitz domain, let $f \in C^0(S; \mathbb{R}^P)$ be finitely developable and let $\delta > 0$ be small enough. Assume that there exist $N \in \mathbb{N}$ and subdomains $V_1, ..., V_N \subset S$ such that*

$$S \cap \partial V_k \subset S \setminus \hat{C}_f \text{ and } V_k \text{ satisfies condition } (B_f) \tag{134}$$

for all $k = 1, ..., N$, and such that

$$S^f_\delta \subset \bigcup_{k=1}^{N} \overline{V}_k \text{ and } V_k \cap V_j = \emptyset \text{ if } k \neq j. \tag{135}$$

Then there exists a subfamily, again denoted $\{V_k\}_{k=1}^N$, such that the following is true:
The set

$$W := \operatorname{int}\left(\bigcup_{k=1}^{N} \bar{V}_k\right) \tag{136}$$

is a subdomain of S, *and*
$$S_\delta^f \subset W; \tag{137}$$
in addition,
$$S \cap \partial W \subset S \setminus \hat{C}_f \text{ and } W \text{ satisfies } (B_f)$$
with
$$S \cap \partial W \subset \bigcup_{i=0}^{N_S} A_{ii}^f. \tag{138}$$
We have
$$S = W \cup \left(\bigcup_{x \in Z_W} (S \cap \overline{S_x^2}) \right), \tag{139}$$
with disjoint unions on the right-hand side.
Moreover,
$$j \neq k \text{ and } x \in \bar{V}_j \cap \bar{V}_k \cap S \implies x \in S_\delta^f \setminus \hat{C}_f; \tag{140}$$
in particular, $\mathcal{H}^1([x]) \geq \delta$. *In addition,*
$$\bar{W} \cap \overline{S}_x^2 \cap S = [x] \text{ for all } x \in Z_W$$
and
$$\bar{S}_x^2 \cap \bar{S}_y^2 \cap S = \emptyset \text{ if } x, y \in Z_W \text{ with } x \neq y. \tag{141}$$

Proof. We write S_δ instead of S_δ^f. By finite developability the set $\hat{C}_f$ consists of finitely many connected components $U_1, ..., U_L$. Setting
$$\delta_0 := \frac{1}{2} \min_{i=1,\ldots,L} \sup \operatorname{dist}_{\partial S}(U_i),$$
by definition of S_δ we have
$$\hat{C}_f \subset S_\delta \tag{142}$$
for all $\delta \in (0, \delta_0)$. We assume without loss of generality that $\delta \in (0, \delta_0)$ is so small that the conclusions of Lemma 7 hold. By deleting those V_k whose closure does not intersect S_δ, we may assume without loss of generality that
$$V_k \cap S_\delta \neq \emptyset \text{ for all } k = 1, ..., N, \tag{143}$$
since $\bar{V}_k \cap S_\delta \neq \emptyset$ is equivalent to $V_k \cap S_\delta \neq \emptyset$ because S_δ is open by Lemma 7. Clearly (135) remains valid.
Define the set W as in (136). Observe that $W \subset S$. In fact, the closure of W is contained in $\bar{V}_1 \cup ... \cup \bar{V}_N$, which in turn is contained in the closure of S. Thus indeed
$$W \subset \operatorname{int} \bar{W} \subset \operatorname{int} \bar{S} = S$$
because S is a Lipschitz domain.
Let us now prove that W is connected. The set
$$W' := S_\delta \cup \bigcup_{k=1}^{N} V_k$$
is connected because $V_k \cap S_\delta \neq \emptyset$ for all $k = 1, ..., N$ and because S_δ and each V_k are connected. By openness of S_δ, the inclusion in (135) implies (137). Thus
$$W' \subset W \subset \bigcup_{k=1}^{N} \bar{V}_k = \bar{W}'.$$

Hence Theorem IV.1.2 in [17] implies that W is connected because W' is connected. Now set $W_1 := V_1$. Then W_1 satisfies condition (B_f). Set $i_1 := 1$. For $j = 1, ..., N-1$ inductively set

$$W_{j+1} := \mathrm{int}(\bar{W}_j \cup \bar{V}_{i_{j+1}}), \text{ where } i_{j+1} \in \{1, ..., N\} \setminus \{i_1, ..., i_j\}$$

is chosen such that $\bar{V}_{i_{j+1}}$ intersects $\bar{W}_j$. Such i_{j+1} is easily seen to exists while $j < N$ because W is connected. By Lemma 5 (viii) each W_j with $j \in \{1, ..., N\}$ satisfies condition (B_f) since $V_{i_{j+1}}$ satisfies (B_f) for all $j < N$. Since $W = W_N$, we conclude that W satisfies condition (B_f).
We claim that

$$k \neq j \text{ and } x \in \bar{V}_k \cap \bar{V}_j \cap S \implies [x] \subset \partial V_k \cap \partial V_j \cap S_\delta. \tag{144}$$

In fact, if $k \neq j$ and $x \in \bar{V}_k \cap \bar{V}_j \cap S$ then $x \in \partial V_k \cap \partial V_j$ because $V_j \cap V_k = \emptyset$ by hypothesis. In particular, $x \in S \setminus \hat{C}_f$, so $[x]$ is well defined. By (B_f) we have

$$[x] \subset \partial V_k \cap \partial V_j.$$

If $x \notin S_\delta$ then by Lemma 7 there would be $i \in \{0, ..., N_S\}$ such that $x \in A_{ii}$. So by Lemma 16 the set $S \setminus [x]$ consists of precisely two connected components S_x^- and S_x^+. And

$$S \cap \partial S_x^- \cap \partial S_x^+ = [x].$$

Since $[x] \subset \partial V_j \cap \partial V_k$ and since $V_j \cap V_k = \emptyset$, after possibly swapping $+$ and $-$, we have $V_j \subset S_x^-$ and $V_k \subset S_x^+$. In fact, since $V_j, V_k \subset S$ are nonempty and open, and since $S \setminus [x]$ consists of exactly two components S_x^- and S_x^+, after possibly swapping $+$ and $-$ we have $V_j \cap S_x^- \neq \emptyset$. Now V_j is connected and does not intersect $\partial S_x^- \subset [x] \cup \partial S$ because $[x] \subset \partial V_j$ and V_j is open. Thus $V_j \subset S_x^-$. Arguing similarly for V_k, we conclude that either $V_k \subset S_x^-$ or $V_k \subset S_x^+$. By Lemma 5 (viii), the set $V_j \cup V_k \cup [x]$ is open, so it contains $B_r(x)$ for small enough r. But since $[x] \subset \partial S_x^- \cap \partial S_x^+$ by Lemma 16, the ball $B_r(x)$ also intersects S_x^+. Since $(V_j \cup [x]) \cap S_x^+ = \emptyset$, indeed we must have $V_k \cap S_x^+ \neq \emptyset$.
But $S_\delta \cap (\partial S_x^+ \cup \partial S_x^-) \subset S_\delta \cap ([x] \cup \partial S) = \emptyset$ because $[x]$ does not intersect S_δ; this follows from (58) since $x \notin S_\delta$. Hence by connectedness either $S_\delta \subset S_x^-$ or $S_\delta \subset S_x^+$. So V_k or V_j does not intersect S_δ. This contradicts (142) or (143). We conclude that $x \in S_\delta$. Hence (144) follows from (58).

And (144) implies (140). By (140) and by the definition of S_δ we have $\mathcal{H}^1([x]) \geq \delta$ whenever $x \in S \cap \bar{V}_k \cap \bar{V}_j$ for some $j \neq k$.
Since W satisfies condition (B_f), Lemma 5 furnishes a countable set Z_W such that

$$S \cap \partial W = \bigcup_{x \in Z_W} [x].$$

Since $S_\delta \subset W$ and W is open, we have $S \cap \partial W \subset S \setminus S_\delta$. Hence (138) follows from Lemma 7. So

$$S \setminus \bigcup_{x \in Z_W} [x] = W \cup \bigcup_{x \in Z_W} S_x^2 \tag{145}$$

by (27) in Lemma 5 (vii). As in that lemma, S_x^2 denotes the component of $S \setminus [x]$ that does not intersect W. And (141) follows from Lemma 5 (v). Observe that (145) implies (139).

□

Proofof Theorem 4. We write S_δ instead of S_δ^f. Since $\hat{C}_f$ consists of finitely many connected components, if $x \in \partial \hat{C}_f$ then $x \in \partial U$ for some connected component U of $\hat{C}_f$. Hence by Proposition 7, for all $x \in S \cap \partial \hat{C}_f$ there is $r_x > 0$

such that $B_{r_x}(x) \subset S$ and such that the connected components $B^0(x)$ and $B^1(x)$ of $B_{r_x}(x) \setminus [x]$ (i.e. open half-disks), satisfy $B^1(x) \subset S \setminus \overline{\hat{C}_f}$ and $B^0(x) \subset \hat{C}_f$.
Since $\hat{q}_f$ is an S-ruling on $S \setminus \hat{C}_f$, its restriction to $B_{r_x}(x)$ is a $B_{r_x}(x)$-ruling on $B_{r_x}(x) \setminus \hat{C}_f$ for all $x \in S \cap \partial\hat{C}_f$. And it can clearly be extended to a $B_{r_x}(x)$-ruling $\tilde{q}_f^x$ defined on all of $B_{r_x}(x)$. This is achieved by setting $\tilde{q}_f^x$ constantly equal to $q_f(x)$ on $B^0(x)$. So by Remark 6, for all $x \in S \cap \partial\hat{C}_f$ the $\tilde{q}_f^x$-integral curve $\Gamma_x \in W^{2,\infty}([-T_x, T_x]; B_{r_x}(x))$ with $\Gamma_x(0) = x$ satisfies $B_{\frac{r_x}{8}}(x) \subset [\Gamma_x(-T_x, T_x)]$; here we have set $T_x = r_x/4$. By the second part of Remark 6 we know (after possibly replacing $\tilde{q}_f^x$ by $-\tilde{q}_f^x$) that

$$\Gamma([-T_x, 0)) \subset B_x^0 \subset \hat{C}_f \text{ and } \Gamma((0, T_x]) \subset B_x^1 \subset S \setminus \overline{\hat{C}_f}. \tag{146}$$

On the other hand, for all $x \in S \setminus \overline{\hat{C}_f}$ Remark 6 yields $T_x > 0$ and

$$\Gamma_x \in W^{2,\infty}([-T_x, T_x]; S \setminus \overline{\hat{C}_f})$$

such that $x \in [\Gamma_x(-T_x, T_x)]$. Hence S is covered by the union of $\hat{C}_f$ with the union of all sets $[\Gamma_x(-T_x, T_x)]$ with $x \in S \setminus \overline{\hat{C}_f}$ or with $x \in S \cap \partial\hat{C}_f$.
Set

$$\hat{S}_\delta := \{x \in S : \mathrm{dist}_{\partial S}(x) \geq \delta\}.$$

Since $\hat{S}_\delta$ is compact, there exist finite subsets $F_1 \subset S \cap \partial\hat{C}_f$ and $F_2 \subset S \setminus \overline{\hat{C}_f}$ such that

$$\hat{S}_\delta \subset \bigcup_{x \in F_1 \cup F_2} [\Gamma([-T_x, T_x])] \cup \hat{C}_f. \tag{147}$$

But for all $x \in F_1$ we have

$$[\Gamma([-T_x, T_x])] \cup \hat{C}_f = [\Gamma([0, T_x])] \cup \hat{C}_f$$

by virtue of (146); this also shows that the union on the right-hand side is disjoint. After shifting the parameter interval for each Γ_x with $x \in F_2$ (and changing the value of T_x), we conclude:

$$\hat{S}_\delta \subset \hat{C}_f \cup \bigcup_{x \in F_1 \cup F_2} [\Gamma_x([0, T_x])]. \tag{148}$$

Now if $x \in S_\delta \setminus \hat{C}_f$ then by definition of S_δ there is $x' \in [x] \cap \hat{S}_\delta$. Thus (148) implies that $x' \in [\Gamma_y([0, T_y])]$ for some $y \in F_1 \cup F_2$. But this implies that $[x'] \subset S \cap \overline{[\Gamma_y(0, T_y)]}$, compare e.g. the proof of Proposition 16. We conclude that

$$S_\delta \subset \hat{C}_f \cup \bigcup_{x \in F_1 \cup F_2} \overline{[\Gamma_x(0, T_x)]}. \tag{149}$$

By inductively restricting each curve (and possibly deleting some; we denote the resulting index sets again by F_1 and F_2 and their lengths again by T_x), it is easy to arrange that the sets $[\Gamma_x(0, T_x)]$ become pairwise disjoint, while (149) still holds. Observe that each curve Γ_x satisfies (121) by virtue of Remark 6.
In a final step we subdivide each curve into curves satisfying (133) as follows: Fix one $y \in F_1 \cup F_2$. For all $t \in [0, T_y]$ we define, with $\overline{s}^+_{\Gamma_y}$, $\underline{s}^-_{\Gamma_y}$ as in formula (77),

$$Y(t) := \{\Gamma_y(t) + sN_y(t) : s \in (\underline{s}^-_{\Gamma_y}(t), \overline{s}^+_{\Gamma_y}(t))\}.$$

Set

$$A^+_\delta := \{t \in [0, T_y] : \overline{s}^+_{\Gamma_y}(t) - s^+_{\Gamma_y}(t) \geq \delta\}$$

and

$$A_\delta^- := \{t \in [0, T_y] : s_{\Gamma_y}^-(t) - \underline{s}_{\Gamma_y}^-(t) \geq \delta\}.$$

Clearly $A_\delta^\pm \subset A_{s_{\Gamma_y}^\pm}$, with $A_{s_{\Gamma_y}^\pm}$ as in formula (78). Proposition 16 implies that $[\Gamma_y(0, T_y)]$ satisfies condition (B_f). Applying Lemma 5 we obtain a set $Z_{[\Gamma_y(0,T_y)]}$ with the properties stated there.

Claim #1. For all

$$x \in Z_{[\Gamma_y(0,T_y)]} \setminus ([\Gamma_y(0)] \cup [\Gamma_y(T_y)])$$

with $\mathcal{H}^1([x]) \geq \delta$ there is $t \in A_-^\delta \cup A_+^\delta$ such that $[x] \subset Y(t)$.

In fact, by Proposition 16 the inclusion (122) or (123) holds. If (122) holds and $\mathcal{H}^1([x]) \geq \delta$ then

$$\overline{s}_{\Gamma_y}^+(t) - s_{\Gamma_y}^+(t) \geq \delta$$

because the segment with endpoints $\Gamma_y(t) + s_{\Gamma_y}^+(t)N_y(t)$ and $\Gamma_y(t) + \overline{s}_{\Gamma_y}^+(t)N_y(t)$ contains $[x]$ by (122). The other case is similar, and the claim follows.

Now we subdivide Γ_y as follows. Proposition 15 that the sets $A_\delta^\pm$ are finite, i.e. there exist $L_y \in \mathbb{N}$ and

$$0 < t_1 < t_2 < ... < t_{L_y} < T_y$$

such that

$$A_\delta^+ \cup A_\delta^- \subset \{0, t_1, ..., t_{L_y}, T_y\}.$$

Set $T_y^{(1)} = t_1$ and denote by $\Gamma_y^{(1)}$ the restriction of Γ_y to $[0, T_y^{(1)}]$. Set $T_y^{(2)} = t_2 - t_1$ and denote by $\Gamma_y^{(2)}$ the restriction of $\Gamma_y(\cdot + t_1)$ to $[0, T_y^{(2)}]$, and so on. It follows that whenever $t \in [0, T_y^{(1)}]$ satisfies

$$|\overline{s}_{\Gamma_y^{(1)}}^+(t) - s_{\Gamma_y^{(1)}}^+(t)| \geq \delta \text{ or } |\underline{s}_{\Gamma_y^{(1)}}^-(t) - s_{\Gamma_y^{(1)}}^-(t)| \geq \delta$$

then $t \in \{0, T_y^{(1)}\}$. Similar implications apply to the other subcurves $\Gamma_y^{(2)}$, $\Gamma_y^{(3)}$ etc. Hence (up to the different notation) each of these subcurves satisfies (133); observe that we can take the supremum in (133) because the sets $A_{\delta/2}^\pm$ are finite, too.

The construction is concluded by relabelling the curves: Let $(\Gamma^{(1)}, T_1)$, $(\Gamma^{(2)}, T_2)$, ... be such that

$$\{(\Gamma^{(1)}, T_1), (\Gamma^{(2)}, T_2), ...\} = \{(\Gamma_y^{(i)}, T_y^{(i)}) : y \in F_1 \cup F_2 \text{ and } i \leq L_y\}$$

By finite developability the set $\hat{C}_f$ consists of finitely many connected components $V_1, ..., V_M$. For $k > M$ set

$$V_k := [\Gamma^{(k-M)}(0, T_{k-M})].$$

Since different components of $\hat{C}_f$ are disjoint and since $[\Gamma^{(k)}(0, T_k)]$ does not intersect $\hat{C}_f$ for any k, we have

$$V_k \cap V_j = \emptyset \text{ if } j \neq k.$$

Hence (135) follows from (149). Furthermore, Proposition 7 shows that if $i < j \leq M$ then $\bar{V}_i \cap \bar{V}_j \cap S = \emptyset$. Moreover, each V_k satisfies (134): If $k \leq M$ then by Proposition 7, and if $k > M$ then by Proposition 16. All remaining assertions follow from Proposition 17 by setting $W_0 = W$ and denoting the elements of Z_W by $x_1, x_2, ...$ and setting $W_k = S_{x_k}^2$ for all $k \geq 1$.

Claim #1 and (133) imply: If $x \in S \cap \partial[\Gamma^{(k)}(0, T_k)]$ satisfies $\mathcal{H}^1([x]) \geq \delta$ then there is $t \in \{0, T_k\}$ such that x is contained in the segment

$$\Big(\Gamma^{(k)}(t) + \underline{s}_{\Gamma^{(k)}}^-(t)N^{(k)}(t), \Gamma^{(k)}(t) + \overline{s}_{\Gamma^{(k)}}^+(t)N^{(k)}(t)\Big).$$

Hence the inclusion (132) is a consequence of (129). □

7 Appendix: Some topological facts

Remark 8 *If γ_0, γ_1 are disjoint closed Jordan curves such that γ_1 is contained in the closure of $U_b(\gamma_0)$, then $U_b(\gamma_1)$ is contained in $U_b(\gamma_0)$.*

Proof. Since $U_b(\gamma_1)$ is open, it suffices to prove that $U_b(\gamma_1)$ does not intersect $U_\infty(\gamma_0)$. Notice that $U_\infty(\gamma_0)$ does not intersect

$$\partial U_b(\gamma_1) = \gamma_1 \subset \overline{U_b(\gamma_0)} = U_b(\gamma_0) \cup \gamma_0.$$

Hence if we had $U_b(\gamma_1) \cap U_\infty(\gamma_0) \neq \emptyset$, then by connectedness $U_\infty(\gamma_0)$ would be contained in $U_b(\gamma_1)$. This would contradict the boundedness of $U_b(\gamma_1)$. □

Lemma 16 *Let $S \subset \mathbb{R}^2$ be a continuous domain and let γ, β be disjoint Jordan arcs contained in S. Then the following hold:*

(i) If both endpoints γ^+ and γ^- of γ are contained in the same connected component $\partial_k S$ of ∂S then $S \setminus \gamma$ consists of exactly two connected components S^1_γ and S^2_γ, and $\partial_k S \setminus \{\gamma^+, \gamma^-\}$ consists of exactly two connected components $\partial^1_k S$ and $\partial^2_k S$. Moreover, there exists a partition of $\{0, ..., N_S\} \setminus \{k\} = I_1 \cup I_2$ into two disjoint subsets I_1 and I_2 such that, for $j = 1, 2$ we have:

$$\overline{U_b(\partial_i S)} \subset S^j_\gamma \text{ for all } i \in I_j \setminus \{0\} \tag{150}$$

and

$$\partial S^j_\gamma = \overline{\gamma} \cup \partial^j_k S \cup \bigcup_{i \in I_j} \partial_i S. \tag{151}$$

(ii) If $k, j \in \{0, ..., N_S\}$ with $k \neq j$ and $\gamma^-, \beta^- \in \partial_k S$ and $\gamma^+, \beta^+ \in \partial_j S$ then $S \setminus (\gamma \cup \beta)$ consists of exactly two connected components S_1 and S_2, and $\bar{S}_1 \cap \bar{S}_2 = \bar{\gamma} \cup \bar{\beta}$. Moreover, S_1 and S_2 are continuous domains.

(iii) Let γ_1, $\gamma_2, \gamma_3 \subset S$ be disjoint Jordan arcs such that $\gamma_1^-, \gamma_2^-, \gamma_3^- \in \partial_k S$ and $\gamma_1^+, \gamma_2^+, \gamma_3^+ \in \partial_j S$ for some $j \neq k$. Then $S \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ consists of exactly three connected components.

(iv) If $k \in \{0, ..., N_S\}$ and $\gamma^\pm, \beta^\pm \in \partial_k S$ then $S \setminus (\gamma \cup \beta)$ consists of exactly three connected components. Exactly one of them contains both γ and β in its boundary.

Proof. Let us prove part (i). We give the proof for the case $k \neq 0$; the case $k = 0$ is similar. After relabelling we may assume that $k = 1$. The set $\partial_1 S \setminus \{\gamma^+, \gamma^-\}$ consists of exactly two connected components $\partial^1_1 S$ and $\partial^2_1 S$. Theorem V.11.8 in [17], applied with $D_1 = U_\infty(\partial_1 S)$ implies that $U_\infty(\partial_1 S) \setminus \gamma$ consists of exactly two connected components U_1 and U_2, and

$$\partial U_j = \gamma \cup \partial^j_1 S \text{ for } j = 1, 2. \tag{152}$$

Since $\partial_0 S$ is connected with $\partial_0 S \cap \partial_j U = \emptyset$, we can choose the labels j of U_j and $\partial^j_1 S$ such that $\partial_0 S \subset U_1$ and $\partial_0 S \cap U_2 = \emptyset$. But U_2 intersects $U_b(\partial_0 S)$ because $\partial_1 S \subset U_b(\partial_0 S)$, and $\partial_1 S \cap \partial U_2 \neq \emptyset$ by (152). Thus by connectedness of U_2 we conclude that

$$U_2 \subset U_b(\partial_0 S). \tag{153}$$

By (152), by connectedness of $\overline{U_b(\partial_i S)}$ and since $\overline{U_b(\partial_i S)} \cap (\gamma \cup \partial_1 S) = \emptyset$ for all $i = 2, ..., N_S$, we conclude that there is a partition $\{2, ..., N_S\} = I_1 \cup I_2$ such that, for $j = 1, 2$,

$$\overline{U_b(\partial_i S)} \subset U_j \text{ for all } i \in I_j. \tag{154}$$

Thus $\overline{U_b(\partial_i S)} \cap U_1 = \emptyset$ for all $i \in I_2$ and $\overline{U_b(\partial_i S)} \cap U_2 = \emptyset$ for all $i \in I_1$. Hence

$$U_j \subset \bigcap_{i \in I_{j'}} U_\infty(\partial_i S), \tag{155}$$

where $j' = 1$ if $j = 2$ and $j' = 2$ if $j = 1$. For $j = 1, 2$ we define

$$S_\gamma^j := U_b(\partial_0 S) \cap U_j \cap \bigcap_{i \in I_j} U_\infty(\partial_i S). \tag{156}$$

Recall (18) and that $U_\infty(\partial_1 S) \setminus \gamma = U_1 \cup U_2$. By (155) we have

$$U_j \cap \bigcap_{i=2}^{N_S} U_\infty(\partial_i S) = U_j \cap \bigcap_{i \in I_j} U_\infty(\partial_i S)$$

for $j = 1, 2$. So we conclude

$$S \setminus \gamma = U_b(\partial_0 S) \cap \big(U_\infty(\partial_1 S) \setminus \gamma\big) \cap \bigcap_{i=2}^{N_S} U_\infty(\partial_i S) = S_\gamma^1 \cup S_\gamma^2.$$

By (153) we have $U_b(\partial_0 S) \cap U_2 = U_2$, so in fact

$$S_\gamma^2 = U_2 \cap \bigcap_{i \in I_2} U_\infty(\partial_i S).$$

Now (i) follows from (156), (154) and (152).

To prove (ii) let us assume that $k \neq 0 \neq j$; the case $k = 0$ is similar. We apply Exercise 3 in Section V.11 of [17] with $J_1 = \partial_k S$, $J_2 = \partial_j S$ and $L_1 = \gamma$, $L_2 = \beta$ and

$$D = U_\infty(\partial_k S) \cap U_\infty(\partial_j S),$$

to find that $D \setminus (\beta \cup \gamma)$ consists of two components D_1 and D_2, each of which is a Jordan domain. Clearly

$$\partial D_m \subset \partial_k S \cup \partial_j S \cup \gamma \cup \beta$$

for $m = 1, 2$. Since $D \setminus (\gamma \cup \beta) = D_1 \cup D_2$, from (18) we have $S \setminus (\gamma \cup \beta) = S_1 \cup S_2$, where

$$S_m := U_b(\partial_0 S) \cap D_m \cap \bigcap_{i \in \{1, \ldots, N_S\} \setminus \{j,k\}} U_\infty(\partial_i S).$$

We have $\partial D_m \subset U_b(\partial_0 S)$ since

$$\partial D_m \subset \partial_k S \cup \partial_j S \cup \gamma \cup \beta.$$

If $D_m = U_\infty(\partial D_m)$, then

$$S_m = U_b(\partial_0 S) \cap U_\infty(\partial D_m) \bigcap U_\infty(\partial_i S),$$

so S_m is a continuous domain. If $D_m = U_b(\partial D_m)$, then $U_b(\partial D_m)$ is contained in $U_b(\partial_0 S)$ by Remark 8 because $\partial D_m \subset U_b(\partial_0 S)$. Hence in this case

$$S_m = U_b(\partial D_m) \cap \bigcap U_\infty(\partial_i S),$$

which is a continuous domain as well. Carrying out Exercise 3 in [17] Section V.11 (we leave this to the reader), one finds that $\bar{D}_1 \cap \bar{D}_2 = \bar{\gamma} \cup \bar{\beta}$. Thus $\bar{S}_1 \cap \bar{S}_2 = \bar{\gamma} \cup \bar{\beta}$ because

$$\bar{\gamma} \cup \bar{\beta} \subset \partial_j S \cup \partial_k S \cup \gamma \cup \beta,$$

and the latter set is contained in

$$U_b(\partial_0 S) \cap \bigcap_{i \neq 0,j,k} U_\infty(\partial_i S).$$

This completes the proof of (ii).

Let us prove (iii). By (ii) $S \setminus (\gamma_1 \cup \gamma_2)$ consists of exactly two components U_3 and V_3. Since γ_3 is connected and

$$\partial U_3 \cup \partial V_3 \subset \gamma_1 \cup \gamma_2 \cup \partial S,$$

we have

$$\gamma_3 \cap (\partial U_3 \cup \partial V_3) = \emptyset.$$

So γ_3 is contained either in V_3 or in U_3. Say $\gamma_3 \subset V_3$, so $\gamma_3 \cap U_3 = \emptyset$. Thus U_3 is a component of $S \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$. (This is an easy exercise; see e.g. the analogous statement in the proof of Lemma 17, where the proof is given.) By (ii), V_3 is a continuous domain, so by (i) the set $V_3 \setminus \gamma_3$ consists of exactly two components V_3' and V_3''. Thus U_3, V_3' and V_3'' are the three connected components of $S \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$.

To prove (iv) we apply (i) to conclude that $S \setminus \gamma$ consists of exactly two components S^1_γ and S^2_γ. Since β is connected and does not intersect $\partial S^1_\gamma \cup \partial S^2_\gamma$, it is contained in one of the two. Say $\beta \in S^1_\gamma$. So $\beta \cap \bar{S}^2_\gamma = \emptyset$. As in the proof of (iii), S^2_γ is therefore a component of $S \setminus (\gamma \cup \beta)$. By (i), $S^1_\gamma \setminus \beta$ consists of exactly two components. Thus $S \setminus (\gamma \cup \beta)$ consists of exactly three components.
Repeating the argument with β and γ swapped, we find one component S^2_β of $S \setminus \beta$ which is also a component of $S \setminus (\gamma \cup \beta)$, and $\gamma \cap \bar{S}^2_\beta = \emptyset$. In particular, $S^2_\gamma \neq S^2_\beta$. So we have identified two components of $S \setminus (\gamma \cup \beta)$ which do not contain $\gamma \cup \beta$ in their closure. Hence there can be at most one component which does.
Since in what follows we will only make use of uniqueness, we leave it to the reader to prove that the third component of $S \setminus (\gamma \cup \beta)$ contains $\gamma \cup \beta$ in its closure. □

Lemma 17 *Let $S \subset \mathbb{R}^2$ be a continuous domain and let Γ_+, Γ_- be disjoint and connected subarcs of ∂S. Assume that $\gamma_1, \gamma_2, \gamma_3 \subset S$ are disjoint Jordan curves with $\gamma_k^+ \in \Gamma_+$ and $\gamma_k^- \in \Gamma_-$ ($k = 1, 2, 3$). Set*

$$V_m = \mathcal{C}(S \setminus (\gamma_k \cup \gamma_l); \gamma_m),$$

where $m = 1, 2, 3$ and (k, l, m) is a permutation of $(1, 2, 3)$. Then $V_1 \cap V_2 \cap V_3 = \emptyset$.

Remark. The claim is false if the $\Gamma_\pm$ are not disjoint: Consider e.g. $S = B_1(0)$ and γ_k the line segments with endpoints given in polar coordinates (r, θ) by $\gamma_k^\pm = (1, \theta_k^\pm)$ with $\theta_1^\pm = \pm\frac{\pi}{6}$, $\theta_2^\pm = \frac{\pi}{2} \pm \frac{\pi}{6}$ and $\theta_3^\pm = \pi \pm \frac{\pi}{6}$.
Proof. First of all notice that the V_m are well defined: By Lemma 16 (ii) and (iv), the set $S \setminus (\gamma_1 \cup \gamma_2)$ consists of at least two connected components. Since γ_3 is connected and does not intersect $\partial S \cup \gamma_1 \cup \gamma_2$, it is contained in one component of $S \setminus (\gamma_1 \cup \gamma_2)$. Thus V_3 is well defined, and so are V_1 and V_2 by analogous arguments. If $\Gamma_+ \subset \partial_i S$ for some $i \neq 0$ then pick $p \in U_b(\partial_i S)$ and define a homeomorphism Ψ_p of $\mathbb{R}^2 \setminus \{p\}$ onto $\mathbb{R}^2 \setminus \{0\}$ by setting $\Psi_p(x) = \frac{x-p}{|x-p|^2}$ for $x \in \mathbb{R}^2 \setminus \{p\}$. Then $\Psi_p(\partial_k S)$, where $k = 0, ..., N_S$, are closed Jordan curves. Clearly,

$$\Psi_p(U_b(\partial_0 S)) = U_\infty(\Psi_p(\partial_0 S)) \tag{157}$$

$$\Psi_p(U_\infty(\partial_i S)) = U_b(\Psi_p(\partial_i S)). \tag{158}$$

Similarly, $\Psi_p(U_\infty(\partial_k S)) = U_\infty(\Psi_p(\partial_k S))$ if $k \notin \{0, i\}$. To prove (158), notice that by continuity $\Psi_p(U_\infty(\partial_i S))$ is connected, that it is bounded because p has positive

distance from $U_\infty(\partial_i S)$, and that its boundary agrees with $\Psi_p(\partial_i S)$ because Ψ_p is a homeomorphism. Keeping in mind that $p \in U_b(\partial_k S)$ if and only if $k \in \{0, i\}$, the other equalities are proven similarly.
By (18) and since Ψ_p is a homeomorphism, the above equalities imply

$$\Psi_p(S) = U_b\big(\Psi_p(\partial_i S)\big) \cap \bigcap_{k\in\{0,\dots,N_S\}\setminus\{i\}} U_\infty\big(\Psi_p(\partial_k S)\big).$$

Thus $\Psi_p(S)$ is a continuous domain with $\partial_0(\Psi_p(S)) = \Psi_p(\partial_i S)$. To save notation, we summarize this argument by assuming without loss of generality that $\Gamma_+ \subset \partial_0 S$. Now, after possibly relabelling the $\partial_i S$, two cases may occur: Either (a) $\Gamma_- \subset \partial_1 S$ and $\Gamma_+ \subset \partial_0 S$ or (b) $\Gamma_- \cup \Gamma_+ \subset \partial_0 S$. Let us first assume (a). Set

$$\tilde{V}_m = \mathcal{C}(\mathbb{R}^2 \setminus (\partial_1 S \cup \partial_0 S \cup \gamma_k \cup \gamma_l); \gamma_m),$$

where (k, l, m) is a permutation of $(1, 2, 3)$. If we prove that $\tilde{V}_1 \cap \tilde{V}_2 \cap \tilde{V}_3 = \emptyset$ then we are done, since $V_m \subset \tilde{V}_m$ because $\partial_0 S \cup \partial_1 S \subset \partial S$. To save notation we summarize this observation by assuming without loss of generality that $\partial S = \partial_0 S \cup \partial_1 S$.
So $S = U_b(\partial_0 S) \cap U_\infty(\partial_1 S)$ and each of the disjoint Jordan arcs γ_1, γ_2, γ_3 lies in S and has one endpoint in $\partial_0 S$ and one in $\partial_1 S$. By Lemma 16, the set $S \setminus (\gamma_i \cup \gamma_j)$ $(i \neq j)$ consists of precisely two connected components. By definition, if (i, j, k) is a permutation of $(1, 2, 3)$ then one of them is V_k. The other one is denoted by U_k. Since $\bar{U}_k \cap V_k = \emptyset$, we have $\gamma_k \cap \bar{U}_k = \emptyset$.
Now U_3 is a component of $S\setminus(\gamma_1\cup\gamma_2\cup\gamma_3)$ since it is a component of $S\setminus(\gamma_1\cup\gamma_2)$ with $U_3\cap\gamma_3 = \emptyset$. Indeed, if W is a component of $S\setminus(\gamma_1\cup\gamma_2\cup\gamma_3)$ with $W\cap U_3 \neq \emptyset$ (such W exists by openness of U_3) then $W \subset U_3$ since W is connected and $W \cap \partial U_3 = \emptyset$ because $\partial U_3 \subset \partial S \cup \gamma_1 \cup \gamma_2$. On the other hand, $U_3 \subset W$ since U_3 is connected and $U_3 \cap \partial W = \emptyset$ because $\partial W \subset \partial S \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$. Thus $W = U_3$, as claimed.
On the other hand, since U_3 is a component of $S \setminus (\gamma_1 \cup \gamma_2)$, Lemma 16 (ii) implies $\gamma_1 \cup \gamma_2 \subset \bar{U}_3$. Arguing similarly for $k = 1, 2$ we conclude that, for $k = 1, 2, 3$, we have: U_k is a connected component of $S \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ with $\gamma_k \cap \bar{U}_k = \emptyset$, and $\gamma_i \cup \gamma_j \subset \bar{U}_k$ if $i, j \neq k$. By the last two facts we have $U_k \neq U_j$ if $k \neq j$. Since by Lemma 16 (iii) the set $S \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3)$ consists of exactly three connected components, we conclude

$$S \setminus (\gamma_1 \cup \gamma_2 \cup \gamma_3) = U_1 \cup U_2 \cup U_3. \tag{159}$$

But $(V_1 \cap V_2 \cap V_3) \cap U_k \subset V_k \cap U_k = \emptyset$ for $k = 1, 2, 3$. Thus $V_1 \cap V_2 \cap V_3$ must be empty by (159) and since by openness it is clearly not a subset of $\gamma_1 \cup \gamma_2 \cup \gamma_3$. This concludes the proof for case (a).
Now assume that (b) holds. Then by connectedness of $U_\infty(\partial_0 S)$ and since every point on $\partial_0 S$ is accessible from $U_\infty(\partial_0 S)$, there exist Jordan arcs $\tilde{\Gamma}_\pm \subset U_\infty(\partial_0 S)$ with the same endpoints as $\Gamma_\pm$. (This follows from corollary to Theorem VI.14.6 in [17].) Since $\gamma_i \subset S \subset U_b(\partial_0 S)$, we have $\gamma_i \cap \tilde{\Gamma}_\pm = \emptyset$ $(i = 1, 2, 3)$. And since

$$\tilde{\Gamma}_\pm \cap \Gamma_\pm \subset \tilde{\Gamma}_\pm \cap \partial S$$

is also empty, the sets

$$\partial_\pm S := \bar{\Gamma}_\pm \cup \tilde{\Gamma}_\pm$$

are closed Jordan curves. Let $R > 0$ so large that $B_R(0)$ contains $\partial_+ S \cup \partial_- S \cup \partial_0 S$. Then $\tilde{S} := B_R(0) \cap U_\infty(\partial_+ S) \cap U_\infty(\partial_- S)$ is a continuous domain, and $S \subset U_b(\partial_0 S) \subset \tilde{S}$. So $V_m \subset \tilde{V}_m$, where $\tilde{V}_m := \mathcal{C}(\tilde{S} \setminus (\gamma_k \cup \gamma_l); \gamma_m)$. Hence it is enough to prove that $\tilde{V}_1 \cap \tilde{V}_2 \cap \tilde{V}_3$ is empty. To do this we apply the first part of the proof with $\tilde{S}$ instead of S. $\square$

Lemma 18 *If $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain then there is $\delta_0 > 0$ such that $S \setminus \bar{B}_\delta(\partial S)$ is connected for all $\delta \in (0, \delta_0)$.*

Proof. We only sketch the proof. For $\varepsilon > 0$ set $\hat{S}_\varepsilon := S \setminus \bar{B}_\varepsilon(\partial S)$. Let $\beta > 0$ be small and let $\delta \in (0, \frac{1}{4}\beta)$. For $i = 0, ..., N_S$ let $\alpha_i \subset S$ be a closed Jordan curve with $\mathrm{dist}_{\partial_i S}(\alpha_i) \in (2\delta, \frac{1}{2}\beta)$. (Existence of such α_i is established by explicit construction or e.g. by Theorem VI.17.1 in [17].) Set

$$S' := U_b(\alpha_0) \cap \bigcap_{i=1}^{N_S} U_\infty(\alpha_i). \tag{160}$$

For small β this is a continuous domain, so it is connected. Moreover, one readily checks that $\hat{S}_\beta \subset S' \subset \hat{S}_\delta$. Thus we conclude that, for all $\beta > 0$ small enough, there is $\delta \in (0, \beta)$ such that any two points in $\hat{S}_\beta$ can be connected by a continuous curve lying in the larger set $\hat{S}_\delta$.

Therefore, it remains to prove that every point in $\hat{S}_\delta \setminus \hat{S}_\beta$ can be connected to some point in $\hat{S}_\beta$ by a continuous curve lying in $\hat{S}_\delta$. So let $x \in \hat{S}_\delta \setminus \hat{S}_\beta$ and let us assume throughout that β is small enough. Then there is $R > 0$ such that for some choice of coordinates, we locally have $\partial S = \mathrm{graph}\, h|_{(-R,R)}$ for some Lipschitz function h. And S locally agrees with the subgraph of h, so x lies in this subgraph. Writing $z = (z_1, z_2)$, we have

$$\mathrm{dist}_{\mathrm{graph}\, h}(z) \sim |z_2 - h(z_1)| \tag{161}$$

because $\mathrm{graph}\, h$ is contained in a cone with vertex $(z_1, h(z_1))$ and opening angle $2\arctan(\mathrm{Lip}\, h)$. Set $y_2 := \sup\{a < x_2 : \mathrm{dist}_{\mathrm{graph}\, h}(x_1, a) \geq 2\beta\}$. Using (161) one checks that, for β small, $y := (x_1, y_2) \in S$. Thus $y \in \hat{S}_\beta$. Since $x \in \hat{S}_\delta$ the ball $B_\delta(x)$ is contained in the subgraph of h. Hence so is $B_\delta([xy])$. Again, this readily implies that $[xy] \subset \hat{S}_\delta$. Thus $[xy]$ is the sought-for curve.

The details omitted above are straightforward and are left to the interested reader.

□

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